SECTION 8

LIFE ANNUITIES

COMMUTATION FUNCTIONS

In real world, actuaries use commutation functions to simplify calculations. Computer calculations are faster with commutation functions.

$D_x = v^x(l_x)$

$N_x = \sum_{t=0}^{\omega-x} D_{x+t}$

$S_x = \sum_{t=0}^{\omega-x} N_{x+t}$

NOTE

Since 2000, the EA exams rarely tested anything other than Nx and Dx.

COMMUTATION FUNCTIONS

Easy to value streams of payments:

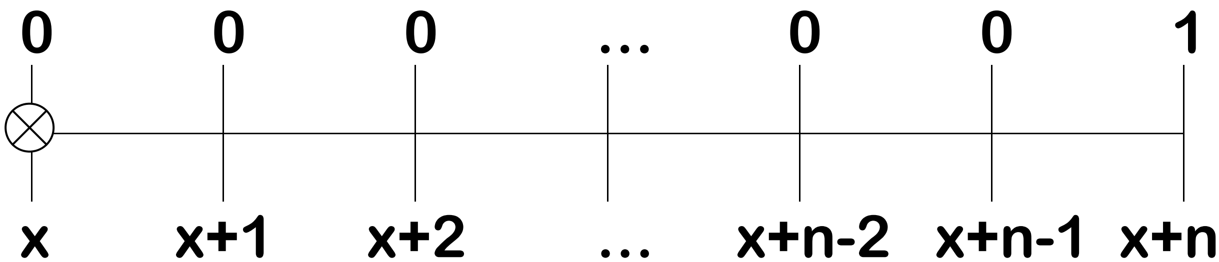
$D_x = v^x(l_x)$
Interpret D_x as single payment upon survival to age x

$N_x = \sum_{t=0}^{\omega-x} D_{x+t}$
Interpret N_x as stream of payments for life, starting at age x

$S_x = \sum_{t=0}^{\omega-x} N_{x+t}$
Interpret S_x as stream of increasing payments for life, starting at age x

ANNUITIES AND ENDOWMENTS

Pure Endowment:
Value of \$1 payable n years from now



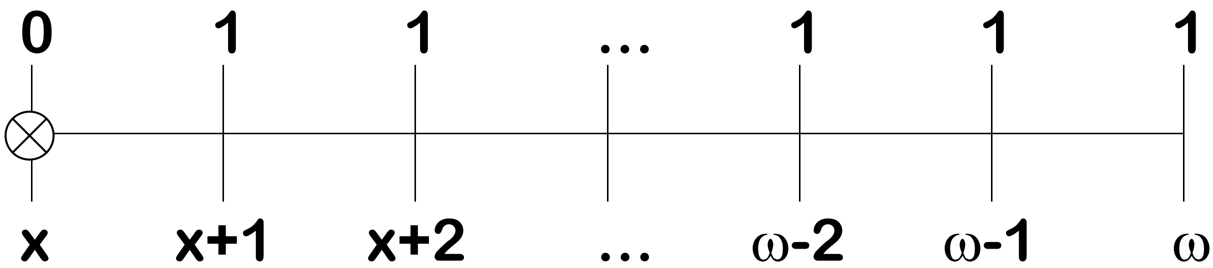
$$PV = {}_{n}E_{x} = \frac{D_{x+n}}{D_{x}} = v^{n}({}_{n}p_{x})$$

Detailed derivation:

$$PV = \frac{D_{x+n}}{D_{x}} = \frac{v^{x+n}(l_{x+n})}{v^{x}(l_{x})} = \frac{v^{n}(l_{x+n})}{l_{x}} = v^{n}({}_{n}p_{x})$$

ANNUITIES AND ENDOWMENTS

Life annuity immediate:
Value of \$1 per annum until death



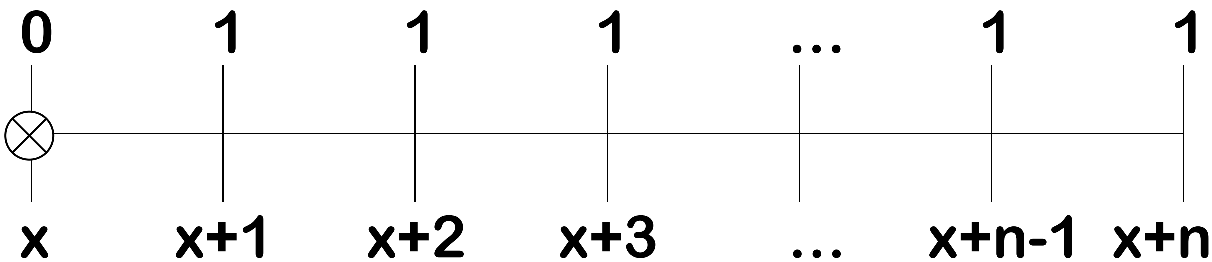
$$PV = a_x$$

$$= \sum_{t=1}^{\omega - x} v^t \, {}_t p_x$$

$$= \frac{N_{x+1}}{D_x}$$

IMMEDIATE LIFE ANNUITIES

Temporary life annuity immediate:
Value of \$1 for next n years, starting at x+1



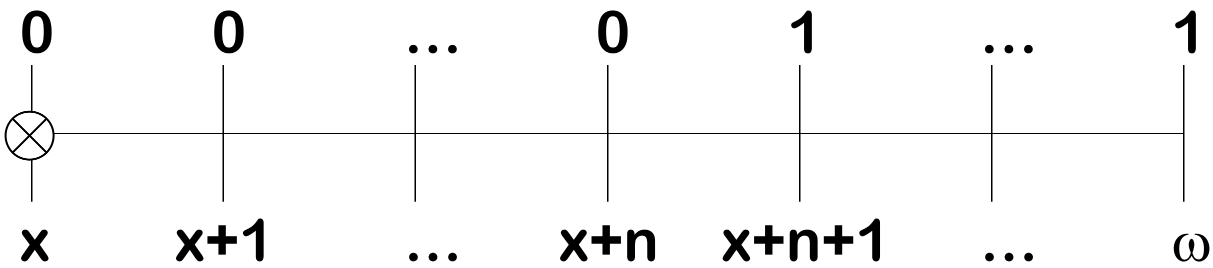
$$PV = a_{x:\overline{n}|}$$

$$= \sum_{t=1}^{n} v^t \, {}_tp_x$$

$$= \frac{N_{x+1} - N_{x+n+1}}{D_x}$$

IMMEDIATE LIFE ANNUITIES

Deferred life annuity immediate:
Value of $1 for life starting after n+1 years



$$PV = {}_{n|}a_x$$

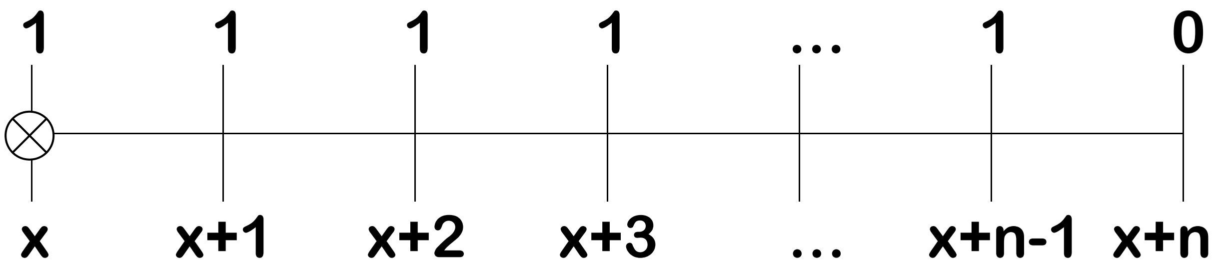
$$= \sum_{t=n+1}^{\omega - x} v^t \, {}_tp_x$$

$$= v^n({}_np_x)\; a_{x+n}$$

$$= \frac{N_{x+n+1}}{D_x}$$

LIFE ANNUITIES DUE

Temporary life annuity due:
Value of $1 for next n years



$$PV = \ddot{a}_{x:\overline{n}|}$$

$$= \sum_{t=0}^{n-1} v^t \, {}_t p_x$$

$$= 1 + a_{x:\overline{n-1}|}$$

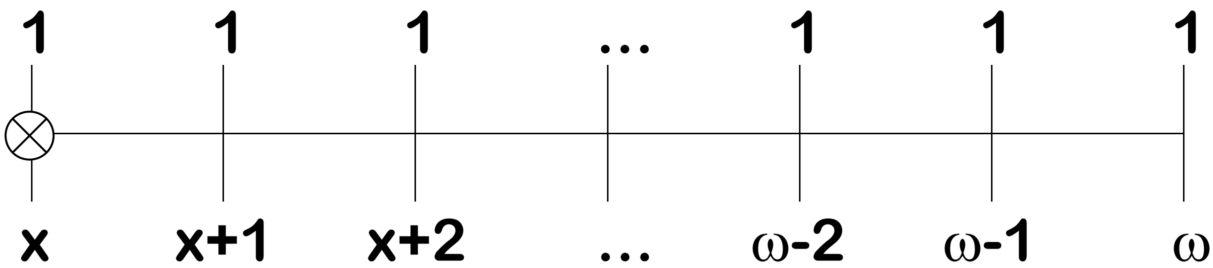
$$= 1 + vp_x(\ddot{a}_{x+1:\overline{n-1}|})$$

$$= 1 + a_{x:\overline{n}|} - {}_nE_x$$

$$= \frac{N_x - N_{x+n}}{D_x}$$

LIFE ANNUITIES DUE

Life annuity due:
Value of \$1 per annum until death



$$PV = \ddot{a}_x$$

$$= \sum_{t=0}^{\omega - x} v^t \, {}_tp_x$$

$$= \frac{N_x}{D_x}$$

$$\ddot{a}_x = 1 + a_x$$

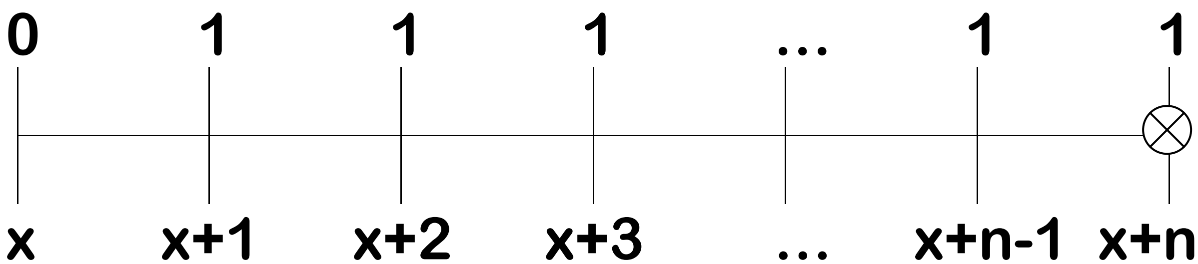
$$= 1 + vp_x \, \ddot{a}_{x+1}$$

$$= \ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x$$

$$a_x = vp_x \, \ddot{a}_{x+1}$$

ACCUMULATED LIFE ANNUITIES

Accumulated temporary life annuity immed: Accumulated value of $1 for n years, starting at age x+1



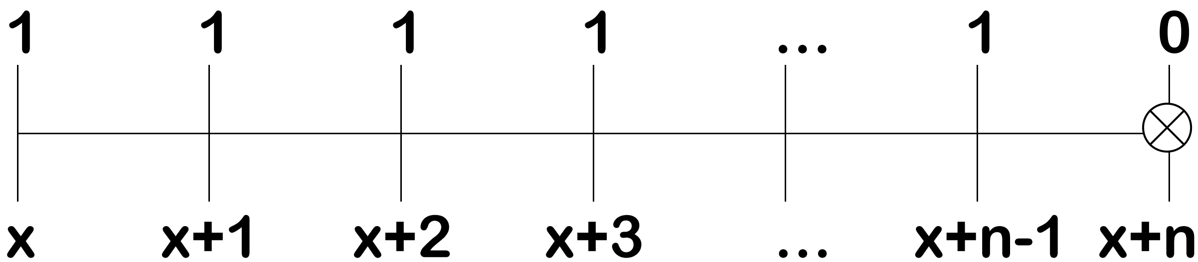
$$FV = s_{x:\overline{n}|}$$

$$= \sum_{t=0}^{n-1} (1+i)^t \, {}_tp_{x+n-t}$$

$$= \frac{N_{x+1} - N_{x+n+1}}{D_{x+n}}$$

ACCUMULATED LIFE ANNUITIES

Accumulated temporary life annuity due: Accumulated value of $1 for n years



$$FV = \ddot{s}_{x:\overline{n}|}$$

$$= \sum_{t=1}^{n} (1+i)^t \, {}_tp_{x+n-t}$$

$$= s_{x:\overline{n+1}|} - 1$$

$$= s_{x-1:\overline{n}|} \, (1+i) \, p_{x+n-1}$$

$$= \frac{N_x - N_{x+n}}{D_{x+n}}$$

EXPECTATION OF LIFE

Curtate expectation of life

e_x is similar to annuity immediate using zero interest rate:

$$
\begin{aligned}
e_x &= {}_1p_x + {}_2p_x + {}_3p_x + \ldots \\
&= \frac{(l_{x+1} + l_{x+2} + l_{x+3} + \ldots)}{l_x}
\end{aligned}
$$

EXPECTATION OF LIFE

Complete expectation of life

$\mathring{e}_x$ allows for survival during year of death

$$\mathring{e}_x = \int_0^\infty {}_t p_x \, dt$$

$$\approx e_x + \tfrac{1}{2}$$

ANNUITIES PAYABLE MORE FREQUENTLY THAN ONCE PER YEAR

$a_x^{(m)}$ is annuity of \$1 per year, payable \$1/m for each 1/mth of the year

Using standard approximations

$$a_x^{(m)} \approx a_x + \frac{(m-1)}{2m}$$

$$\ddot{a}_x^{(m)} = a_x^{(m)} + \frac{1}{m}$$

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{(m-1)}{2m}$$

ANNUITIES PAYABLE MORE FREQUENTLY THAN ONCE PER YEAR

$\ddot{a}^{(m)}_{x:\overline{n}|}$ represents a temporary life annuity of $1 per year, payable $1/m for each 1/mth of the year

Using standard approximations

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \ddot{a}^{(m)}_{x} - {}_{n|}\ddot{a}^{(m)}_{x}$$

$$a^{(m)}_{x:\overline{n}|} = a^{(m)}_{x} - {}_{n|}a^{(m)}_{x}$$

$$a^{(m)}_{x:\overline{n}|} \approx a_{x:\overline{n}|} + \frac{(m-1)}{2m}[1 - {}_{n}E_{x}]$$

PRESENT VALUE CALCULATIONS COMMUTATION FUNCTIONS

Life annuity due of \$1 per year, payable 1/12 at the start of each month

PV = $\ddot{a}_x^{(12)}$

Interpret $N_x^{(12)}$ as stream of monthly payments for life, starting at age x

$$N_x^{(12)} = N_x - \frac{11}{24} D_x$$

$$\ddot{a}_x^{(12)} = N_x^{(12)} / D_x$$

PRESENT VALUE CALCULATIONS COMMUTATION FUNCTIONS

Temporary life annuity due of \$1 per year for n years, payable 1/12 at the start of each month

$$PV = \ddot{a}^{(12)}_{x:\overline{n}|}$$

$$= (N^{(12)}_{x} - N^{(12)}_{x+n})/D_x$$

CONTINUOUS ANNUITIES

Annuity pays \$1 per annum, continuously over each year:

$$\begin{aligned}
\bar{a}_x &= \lim_{m \to \infty} a_x^{(m)} \\
&= \int_0^\infty v^t \, {}_t p_x \, dt \\
&\approx a_x + \frac{1}{2} - \frac{1}{12}(\mu_x + \delta) \\
&\approx a_x + \frac{1}{2}
\end{aligned}$$

ANNUITY CALCULATION

DeMoivre's law

$l_x = \omega - x$

$d_x = 1$ at every age

$p_x = l_{x+1} / l_x$

$$= \frac{\omega - x - 1}{\omega - x}$$

$$a_x = \sum_{t=1}^{\omega - x} v^t \, {}_t p_x$$

Let $n = \omega - x$

$$a_x = \frac{n - \ddot{a}_{\overline{n|}i}}{ni}$$

Derivation shown on next page

ANNUITY CALCULATION

DeMoivre's law

$$a_x = v^1({}_1p_x) + v^2({}_2p_x) + v^3({}_3p_x) + \ldots v^{\omega-x}({}_{\omega-x}p_x)$$

$$= v^1\frac{(\omega-x-1)}{\omega-x} + v^2\frac{(\omega-x-2)}{\omega-x} + \ldots + v^{\omega-x}\frac{(\omega-x-(\omega-x))}{\omega-x}$$

$$= v^1\frac{(\omega-x)}{\omega-x} + v^2\frac{(\omega-x)}{\omega-x} + \ldots + v^{\omega-x}\frac{(\omega-x)}{\omega-x} - \frac{1v^1}{\omega-x}$$

$$- \frac{2v^2}{\omega-x} - \ldots - \frac{(\omega-x)v^{\omega-x}}{\omega-x}$$

Let $n = \omega - x$

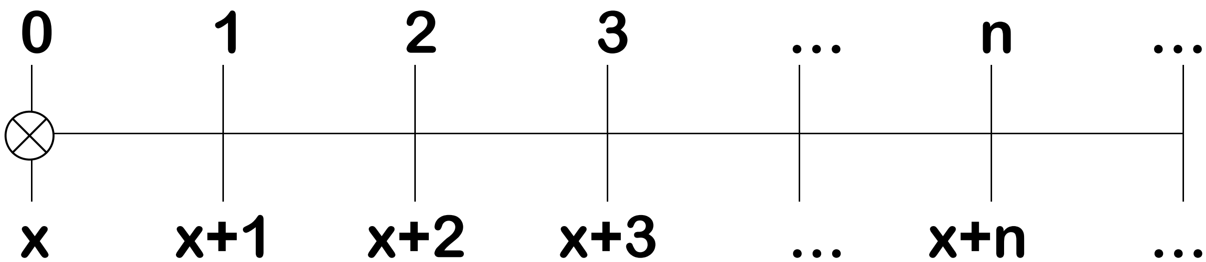
$$a_x = v^1 + v^2 + \ldots + v^n - (1/n)(v^1 + 2v^2 + \ldots + nv^n)$$

$$= a_{\overline{n|}} - (1/n)(Ia)_{\overline{n|}}$$

$$= \frac{n - \ddot{a}_{\overline{n|}i}}{ni}$$

INCREASING ANNUITIES

Increasing immediate life annuity



$$(Ia)_x = \sum_{t=1}^{\omega-x} t v^t \, {}_t p_x$$

$$= \sum_{t=1}^{\omega-x} {}_{t|}a_x$$

Easy to value this with commutation functions:

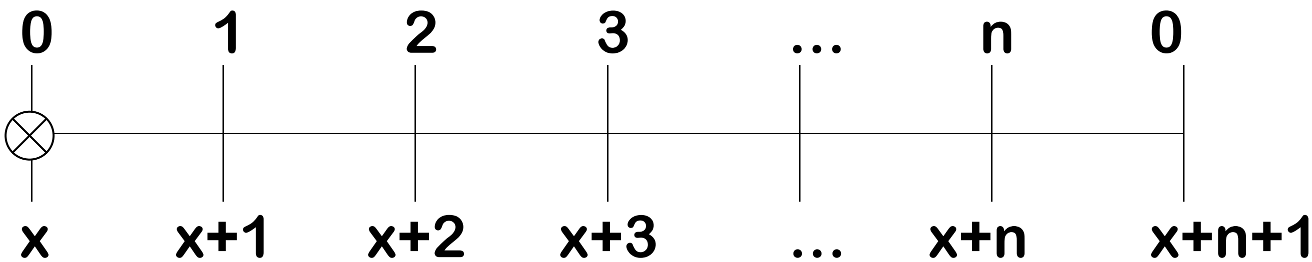
$$(Ia)_x = \sum_{t=1}^{\omega-x} {}_{t|}a_x$$

$$= \frac{(N_{x+1} + N_{x+2} + \ldots)}{D_x}$$

$$= \frac{S_{x+1}}{D_x}$$

INCREASING ANNUITIES

Increasing temporary immediate life annuity



$$(Ia)_{x:\overline{n}|} = a_{x:\overline{n}|} + {}_{1|}a_{x:\overline{n-1}|} + {}_{2|}a_{x:\overline{n-2}|} + \ldots + {}_{n-1|}a_{x:\overline{1}|}$$

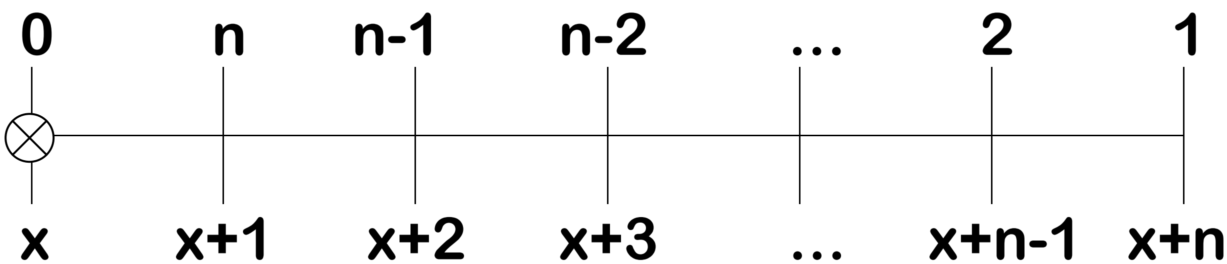
$$= \sum_{t=0}^{n-1} {}_{t|}a_{x:\overline{n-t}|}$$

$$= \frac{N_{x+1} - N_{x+n+1} + N_{x+2} - N_{x+n+1} + \ldots + N_{x+n} - N_{x+n+1}}{D_x}$$

$$= \frac{S_{x+1} - S_{x+n+1} - n^*N_{x+n+1}}{D_x}$$

DECREASING ANNUITIES

Decreasing temporary immediate life annuity



$$(Da)_{x:\overline{n}|} = \sum_{t=1}^{n} a_{x:\overline{t}|}$$

$$= a_{x:\overline{n}|} + a_{x:\overline{n-1}|} + a_{x:\overline{n-2}|} + \ldots + a_{x:\overline{1}|}$$

$$= \frac{N_{x+1} - N_{x+n+1} + N_{x+1} - N_{x+n} + \ldots + N_{x+1} - N_{x+2}}{D_x}$$

$$= \frac{n^*N_{x+1} - (S_{x+2} - S_{x+n+2})}{D_x}$$

KEY CONCEPTS

SECTION VIII - ANNUITIES

1. Commutation functions:
   D_x, N_x, S_x

2. Actuarially equivalent ➔ equal present values

3. Go back to first principles for unusual payment definitions

4. $vp_x \ddot{a}_{x+1} = a_x$

5. Expectation of life, both complete and curtate

6. $\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{(m-1)}{2m}$

7. DeMoivre's law - annuity