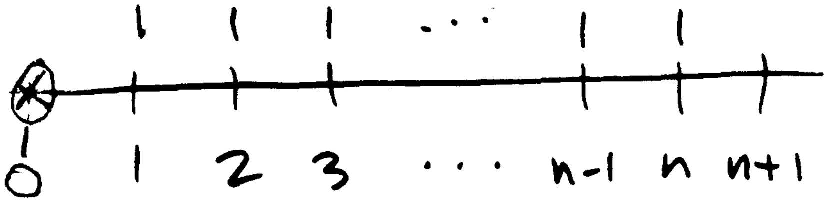
SECTION 3

ANNUITIES

ANNUITY IMMEDIATE



$a_{\overline{n}|i}$ = PV of series of n payments of \$1 valued one period before first payment

$$a_{\overline{n}|} = v + v^2 + v^3 + \ldots + v^{n-1} + v^n$$

$$v\,a_{\overline{n}|} = v^2 + v^3 + \ldots + v^{n-1} + v^n + v^{n+1}$$

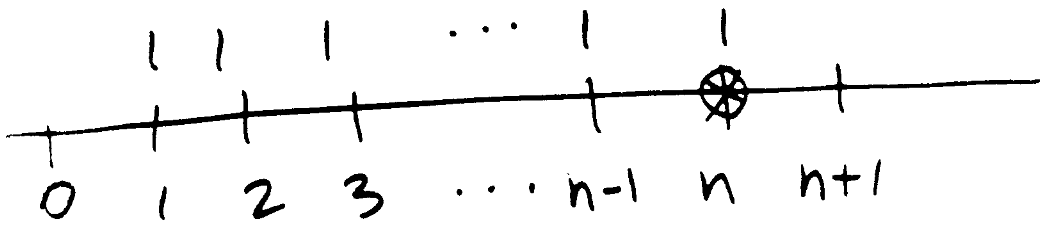
$$(1-v)\,a_{\overline{n}|} = v - v^{n+1}$$

$$a_{\overline{n}|} = (v - v^{n+1})/(1-v)$$

$$= v(1-v^n)/(1-v)$$

$$= \frac{1-v^n}{1+i-1} = \frac{1-v^n}{i}$$

ANNUITY IMMEDIATE



$s_{\overline{n}|i}$ = Accumulated value of series of n payments of \$1 valued <u>at</u> last payment date

$$s_{\overline{n}|} = 1 + (1+i)^1 + (1+i)^2 + \dots + (1+i)^{n-1}$$

$$(1+i)s_{\overline{n}|} = (1+i)^1 + (1+i)^2 + \dots + (1+i)^{n-1} + (1+i)^n$$

$$i\,s_{\overline{n}|} = (1+i)^n - 1$$

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

ANNUITY IMMEDIATE

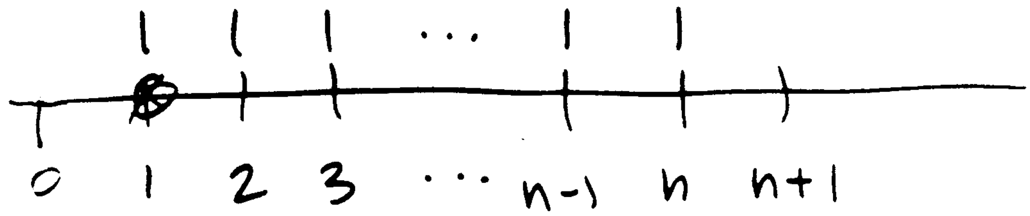
Relationships: $a_{\overline{n}|}$ and $s_{\overline{n}|}$

$$s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}$$

$$1 = i\, a_{\overline{n}|} + v^n \rightarrow \text{Verbal interpretation}$$

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$$

ANNUITY DUE



$\ddot{a}_{\overline{n}|i}$ = PV of series of n payments of \$1 valued at date of first payment

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \ldots + v^{n-1}$$

$$v\ddot{a}_{\overline{n}|} = \quad v + v^2 + \ldots + v^{n-1} + v^n$$

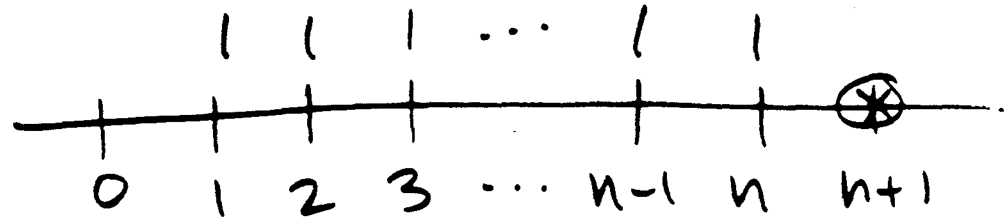
$$(1-v)\ddot{a}_{\overline{n}|} = 1 - v^n$$

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$$

$$= \frac{i}{d}(a_{\overline{n}|}) = (1+i)\,a_{\overline{n}|}$$

$$= 1 + a_{\overline{n-1}|}$$

ANNUITY DUE



$\ddot{s}_{\overline{n}|i}$ = Accomulated value of series of n payments of \$1 valued one period after final payment

$$\ddot{s}_{\overline{n}|} = (1+i)+(1+i)^2+\dots+(1+i)^{n-1}+(1+i)^n$$

$$v\ddot{s}_{\overline{n}|} = 1+(1+i)+(1+i)^2+\dots+(1+i)^{n-1}$$

$$(1-v)\ddot{s}_{\overline{n}|} = (1+i)^n-1$$

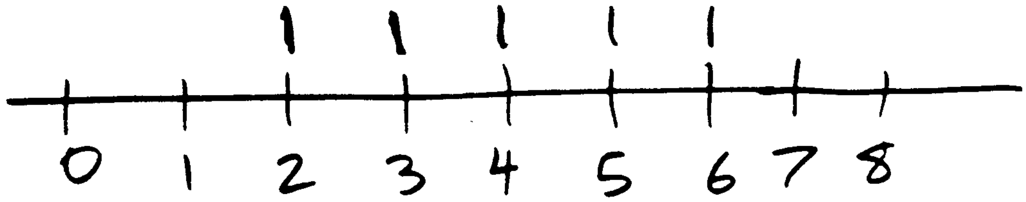
$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n-1}{d}$$

$$= \frac{i}{d}s_{\overline{n}|} = (1+i)s_{\overline{n}|}$$

$$= s_{\overline{n+1}|}-1$$

$$\frac{1}{\ddot{a}_{\overline{n}|}} = \frac{1}{\ddot{s}_{\overline{n}|}}+d$$

VALUE OF PAYMENT STREAM



There are <u>at least</u> two definitions for the payment stream's value at each of the 9 points in time:

0 : $v^2\ddot{a}_{\overline{5}|}$

1 : $a_{\overline{5}|}$

2 : $\ddot{a}_{\overline{5}|}$

3 : $s_{\overline{2}|}+a_{\overline{3}|}$

4 : $s_{\overline{3}|}+a_{\overline{2}|}$

5 : $\ddot{s}_{\overline{3}|}+\ddot{a}_{\overline{2}|}$

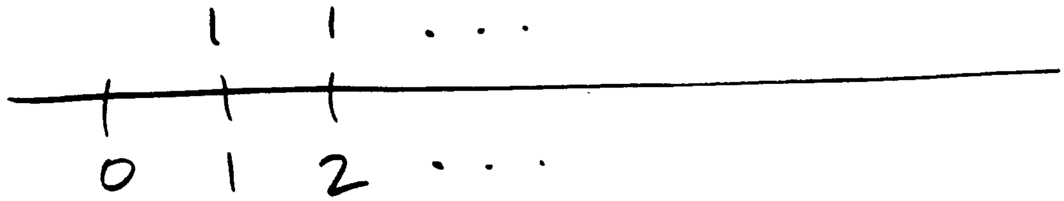
6 : $s_{\overline{5}|}$

7 : $\ddot{s}_{\overline{5}|}$

8 : $(1+i)^2 s_{\overline{5}|}$

HINT: $a_{\overline{x}|} - a_{\overline{y}|}$

PERPETUITIES



$$a_{\overline{\infty}|} = v + v^2 + v^3 + \cdots$$

$$v\,a_{\overline{\infty}|} = v^2 + v^3 + \cdots$$

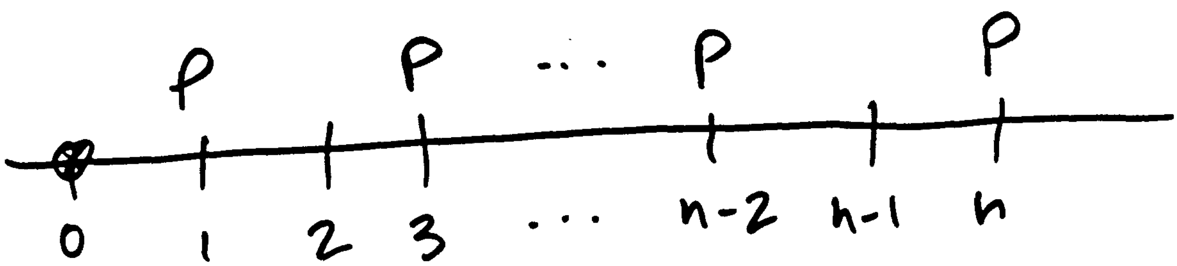
$$(1-v)\,a_{\overline{\infty}|} = v$$

$$a_{\overline{\infty}|} = \frac{v}{1-v} = \frac{1}{i}$$

$$\ddot{a}_{\overline{\infty}|} = (1+i)\,a_{\overline{\infty}|} = \frac{1+i}{i} = \frac{1}{iv} = \frac{1}{d}$$

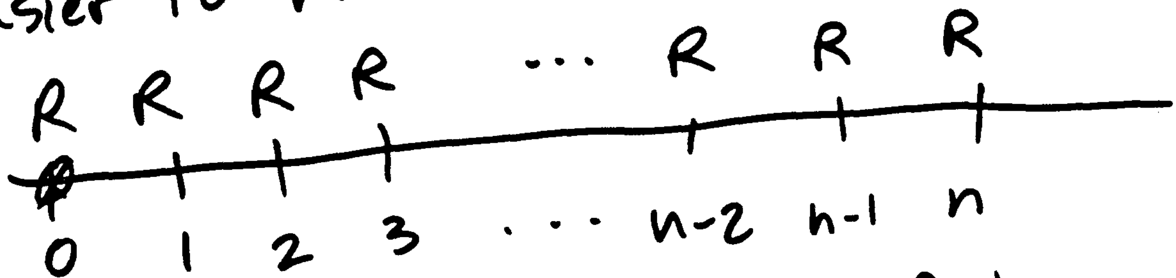
$$a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|} = \lim_{n\to\infty} \frac{1-v^n}{i} = \frac{1}{i}$$

LESS FREQUENT THAN i CONVERTIBLE



Let x = PV of series of payments in amount P every other year

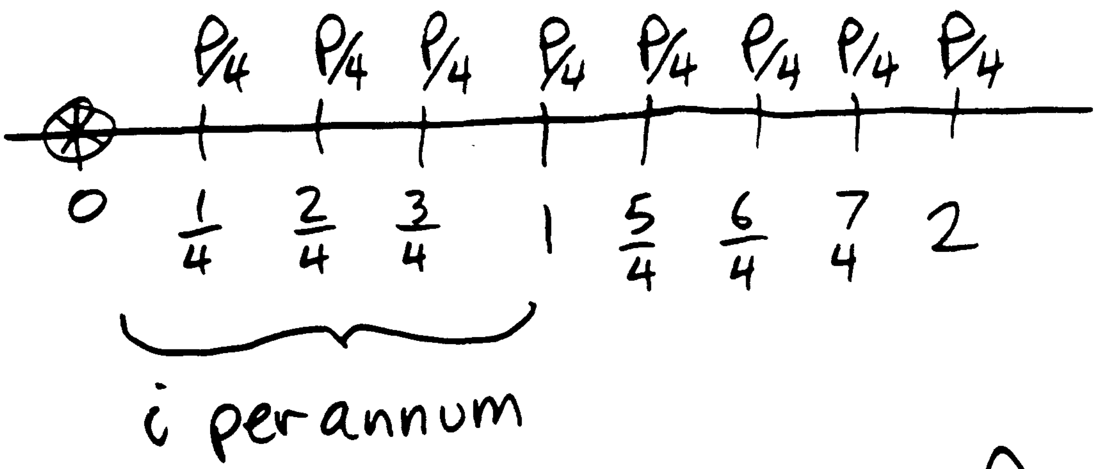
Easier to value this series instead



We replaced each payment P by two payments R such that $P = R \cdot s_{\overline{2}|i}$

$$x = R\,\ddot{a}_{\overline{n+1}|}$$
$$= P\left(\frac{\ddot{a}_{\overline{n+1}|}}{s_{\overline{2}|}}\right)$$

MORE FREQUENT THAN i CONVERTIBLE



X = PV of series of payments P per annum, payable quarterly

$$= P\, a^{(4)}_{\overline{2}|i}$$

$$a^{(m)}_{\overline{n}|} = \frac{1}{m}\left[v^{\frac{1}{m}} + v^{\frac{2}{m}} + \ldots + v^{n-\frac{1}{m}} + v^{n} \right]$$

$$= \frac{1}{m}\left[\frac{v^{\frac{1}{m}} - v^{n+\frac{1}{m}}}{1 - v^{\frac{1}{m}}} \right]$$

$$= \frac{v^{\frac{1}{m}}}{m}\left[\frac{1 - v^{n}}{1 - v^{\frac{1}{m}}} \right]$$

$$= \frac{1 - v^{n}}{i^{(m)}}$$

MORE FREQUENT THAN i CONVERTIBLE

$$a_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}} a_{\overline{n}|}$$

$$s_{\overline{n}|}^{(m)} = a_{\overline{n}|}^{(m)} (1+i)^n = \frac{i}{i^{(m)}} s_{\overline{n}|}$$

$$\frac{1}{a_{\overline{n}|}^{(m)}} = \frac{1}{s_{\overline{n}|}^{(m)}} + i^{(m)}$$

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1-v^n}{d^{(m)}} = \frac{d}{d^{(m)}} \ddot{a}_{\overline{n}|} = \frac{i}{d^{(m)}} a_{\overline{n}|}$$

$$\ddot{s}_{\overline{n}|}^{(m)} = \frac{(1+i)^n - 1}{d^{(m)}} = \frac{d}{d^{(m)}} \ddot{s}_{\overline{n}|} = \frac{i}{d^{(m)}} s_{\overline{n}|}$$

$$\ddot{a}_{\overline{n}|}^{(m)} = (1+i)^{\frac{1}{m}} a_{\overline{n}|}^{(m)}$$

$$\ddot{s}_{\overline{n}|}^{(m)} = (1+i)^{\frac{1}{m}} s_{\overline{n}|}^{(m)}$$

$$\frac{1}{\ddot{a}_{\overline{n}|}^{(m)}} = \frac{1}{\ddot{s}_{\overline{n}|}^{(m)}} + d^{(m)}$$

CONTINUOUS ANNUITIES

$$\bar{a}_{\overline{n}|} = \int_0^n v^t \, dt$$

$$= \left. \frac{v^t}{\ln v} \right]_0^n$$

$$= \frac{1-v^n}{\delta}$$

$$\bar{a}_{\overline{n}|} = \lim_{m \to \infty} a_{\overline{n}|}^{(m)} = \lim_{m \to \infty} \frac{1-v^n}{i^{(m)}} = \frac{1-v^n}{\delta}$$

$$\bar{a}_{\overline{n}|} = \frac{1-e^{-n\delta}}{\delta}$$

$$\bar{s}_{\overline{n}|} = \frac{e^{n\delta}-1}{\delta}$$

PAYMENTS IN ARITHMETIC PROGRESSION

	P	$P+Q$	$P+2Q$		$P+(n-2)Q$	$P+(n-1)Q$
0	1	2	3	$\cdots$	$n-1$	n

$$X = vP + v^2(P+Q) + v^3(P+2Q) + $$
$$v^{n-1}[P+(n-2)Q] + v^n[P+(n-1)Q]$$

$$(1+i)X = P + v(P+Q) + v^2(P+2Q) + $$
$$v^{n-2}[P+(n-2)Q] + v^{n-1}[P+(n-1)Q]$$

$$iX = P + Qv + Qv^2 + \dots + Qv^{n-1} - v^n[P+(n-1)Q]$$
$$= P(1-v^n) + Q[v + v^2 + \dots + v^n] - nQv^n + Qv^n$$
$$= P(1-v^n) + Q a_{\overline{n}|} - nQv^n$$

$$X = P\left(\frac{1-v^n}{i}\right) + Q\left(\frac{a_{\overline{n}|} - nv^n}{i}\right)$$

$$= P\, a_{\overline{n}|i} + Q\left(\frac{a_{\overline{n}|i} - nv^n}{i}\right)$$

Increasing / Decreasing Annuities

$$(Ia)_{\overline{n}|} = a_{\overline{n}|} + 1\left(\frac{a_{\overline{n}|} - nv^n}{i}\right)$$

$$= \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

$$(Is)_{\overline{n}|} = (1+i)^n (Ia)_{\overline{n}|}$$

$$(Da)_{\overline{n}|} = n\,a_{\overline{n}|} - 1\left(\frac{a_{\overline{n}|} - nv^n}{i}\right)$$

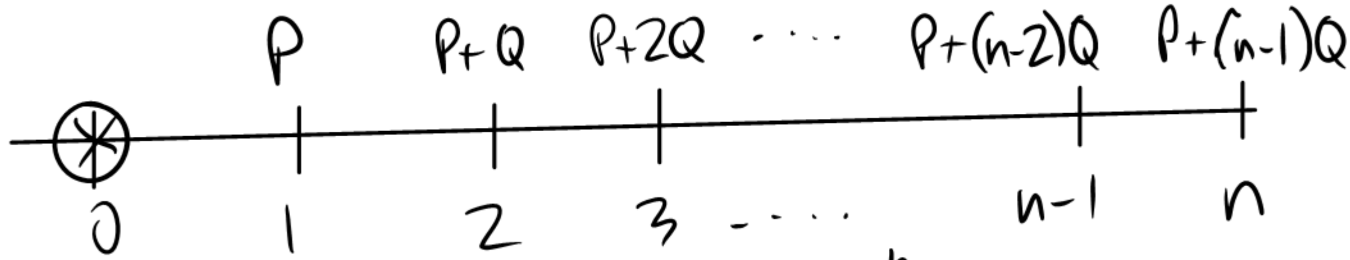
$$= \frac{n - a_{\overline{n}|}}{i}$$

$$(Ds)_{\overline{n}|} = (1+i)^n (Da)_{\overline{n}|}$$

$$(I\ddot{a})_{\overline{n}|} = \frac{i}{d}(Ia)_{\overline{n}|}$$

$$(D\ddot{s})_{\overline{n}|} = \frac{i}{d}(Ds)_{\overline{n}|} = (1+i)^n (D\ddot{a})_{\overline{n}|}$$

INCREASING / DECREASING ANNUITY

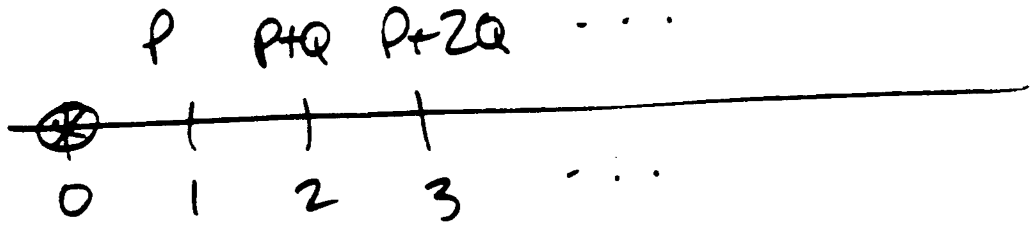


$$X = P a_{\overline{n}|i} + Q\left[\frac{\ddot{a}_{\overline{n}|i} - nv^n}{i}\right]$$

KEY POINT-

No need to memorize formulas on page 44. Just use this general formula instead!

INCREASING PERPETUITY



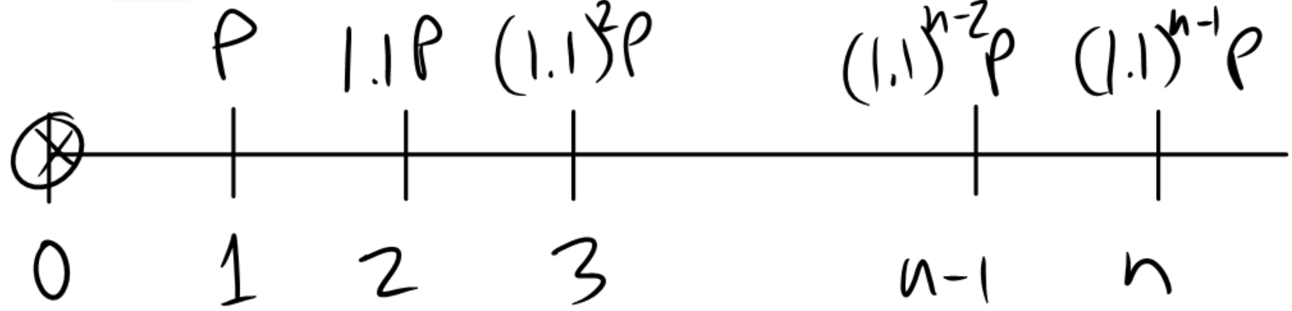
$$X = \lim_{n\to\infty}\left[P a_{\overline{n}|} + Q\left(\frac{a_{\overline{n}|} - nv^n}{i}\right)\right]$$

$$= P\left(\lim_{n\to\infty}\frac{1-v^n}{i}\right) + \frac{Q}{i}\left[\lim_{n\to\infty}\frac{1-v^n}{i} - \lim_{n\to\infty} nv^n\right]$$

$$= \frac{P}{i} + \frac{Q}{i^2}$$

NO DECREASING PERPETUITY

GEOMETRICALLY INCREASING ANNUITY



Annuity immediate with payment of P, increases 10% per year, total of n payments.

$X =$ present value

$$= \frac{P}{(1+i)^1} + \frac{(1.1)P}{(1+i)^2} + \dots + \frac{(1.1)^{n-1}P}{(1+i)^n}$$

$$= \left(\frac{P}{1+i}\right)\left[1 + \frac{1.1}{1+i} + \dots + \frac{(1.1)^{n-1}}{(1+i)^{n-1}}\right]$$

$$= \left(\frac{P}{1+i}\right) \ddot{a}_{\overline{n}|j} \text{ where } 1+j = \frac{1+i}{1.10}$$

Offset rate of increase in payment, value annuity at new interest rate j

KEY CONCEPTS

SECTION III - ANNUITIES

1. Time line diagram

2. Annuity Immediate / Annuity Due

3. First principles approach

4. Arbitrary payment stream

5. Perpetuities

6. Payments more (or less) frequent than interest compounding period

7. Payments in arithmetic progression

8. Geometrically increasing payments