SECTION 7

LIFE CONTINGENCIES

LIFE CONTINGENCIES

Omega - last age in mortality table: ω

Survival function s(x):
Probability of survival from birth to x

$s(0) = 1$

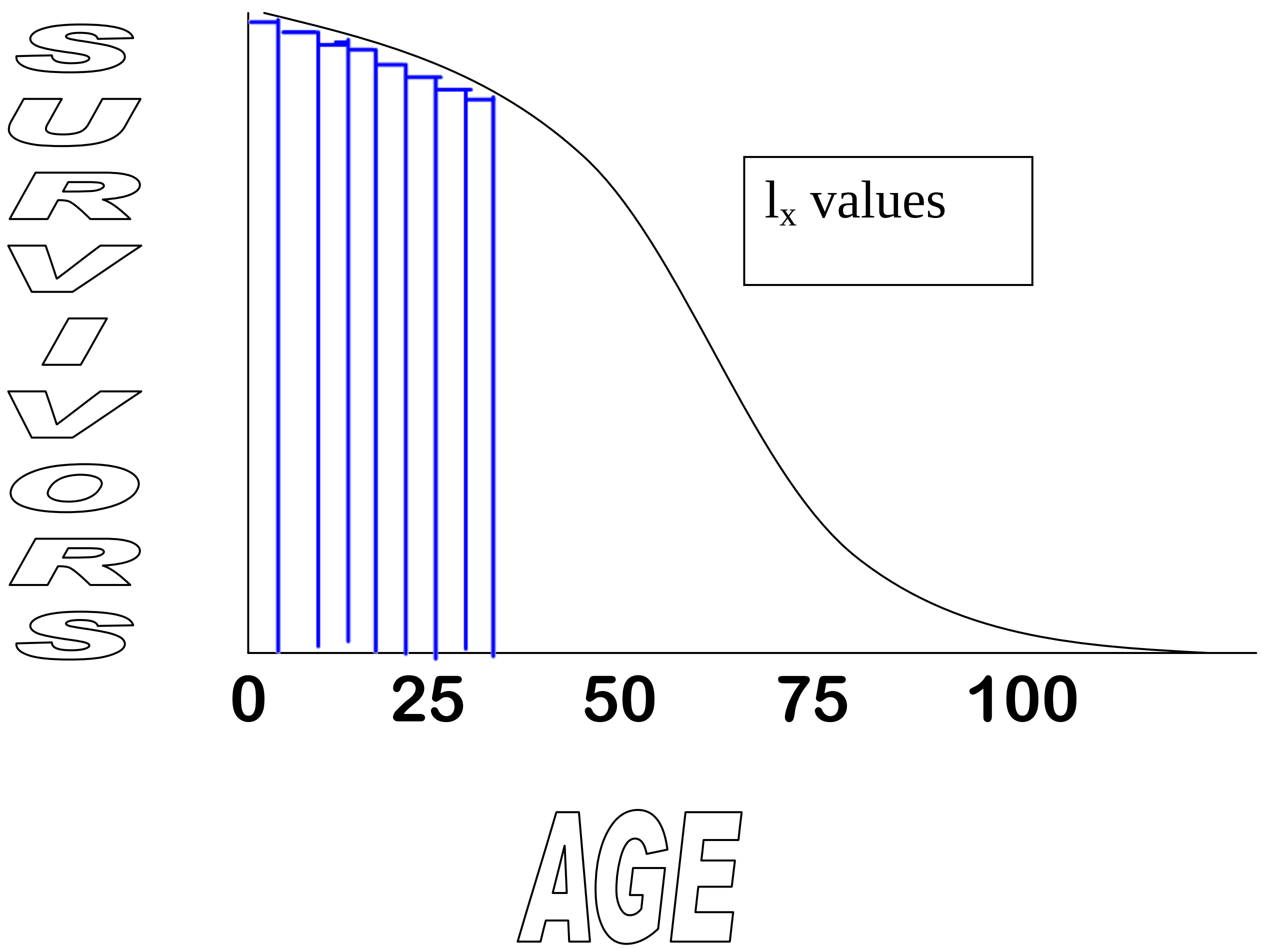
$s(\omega) = 0$

$l_x = K(s(x))$ = # people alive at age x

$K = l_0$ radix of mortality table

LIFE CONTINGENCIES

Graph of l_x curve



FORCE OF MORTALITY

Force of mortality measures mortality at exact age x, expressed as annual rate

$$\mu_x = \frac{-s'(x)}{s(x)}$$ **Bowers 3.2.13**

l_x= K(s(x)) = # people alive at age x

$$\mu_x = \frac{-1}{l_x}\left[\frac{d}{dx} l_x\right]$$

FORCE OF MORTALITY

$$\mu_x = \frac{-s'(x)}{s(x)}$$ Bowers 3.2.13

$$\mu_y = \frac{-d[\ln(s(y))]}{dy}$$

$$-\int_x^{x+n} \mu_y \, dy = \Big[\ln(s(y))\Big]_x^{x+n}$$

$$= \ln[s(x+n)] - \ln[s(x)]$$

$$= \ln[s(x+n) / s(x)]$$

$$= \ln({}_np_x)$$

$${}_np_x = e^{-\int_x^{x+n} \mu_y \, dy}$$ Bowers 3.2.14

LIFE CONTINGENCIES

Probability formulas using l_x

Survive from age x to age x+1
$p_x = l_{x+1} / l_x$

Die between age x and age x+1
$q_x = (l_x - l_{x+1}) / l_x$

Deaths between age x and age x+1
$d_x = l_x - l_{x+1}$

Die between age x and age x+1
$q_x = d_x / l_x$

Survive from age x to age x+n

$_{n}p_{x} = l_{x+n} / l_{x}$

Die between age x and age x+n

$_{n}q_{x} = (l_{x} - l_{x+n}) / l_{x}$

Deaths between age x and age x+n

$_{n}d_{x} = l_{x} - l_{x+n}$

Die between age x and age x+n

$_{n}q_{x} = {}_{n}d_{x} / l_{x}$

Deferred probability of death - live for n years, then die in the next year

$$ {}_{n}|q_{x} = (l_{x+n} - l_{x+n+1}) / l_{x} $$
$$ = d_{x+n} / l_{x} $$

$$ {}_{n}|_{m}q_{x} = {}_{n}p_{x} * {}_{m}q_{x+n} $$
$$ = (l_{x+n} - l_{x+n+m}) / l_{x} $$
$$ = {}_{m}d_{x+n} / l_{x} $$

Survive from age x to age x+n

$$ {}_{n}p_{x} = 1 - {}_{n}q_{x} $$
$$ = (l_{x} - d_{x} - d_{x+1} - \ldots - d_{x+n-1}) / l_{x} $$
$$ = 1 - ({}_{0}|q_{x} + {}_{1}|q_{x} + {}_{2}|q_{x} + \ldots + {}_{n-1}|q_{x}) $$

$$ 1 = {}_{0}|q_{x} + {}_{1}|q_{x} + \ldots {}_{n}|q_{x} + {}_{n+1}|q_{x} + \ldots $$

Probability of surviving n years equals the probability of dying after n years

$$ {}_{n}p_{x} = {}_{n}|q_{x} + {}_{n+1}|q_{x} + {}_{n+2}|q_{x} + \ldots $$

MORTALITY ASSUMPTIONS

Fit formula to entire mortality table:

DeMoivre's law – overly simplified

$\mu_x = (\omega - x)^{-1}$

$l_x = \omega - x$

$d_x = 1$ at every age

EXTRA CREDIT – not tested on exam

Gompertz' law

$\mu_x = Bc^x$

Makeham's law

$\mu_x = A + Bc^x$

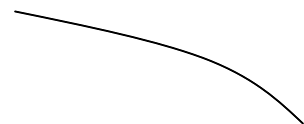
MORTALITY ASSUMPTIONS

Look at the period from age x to age x+1

Constant force of mortality ➔

$\mu_{x+t} = \mu \qquad 0 \leq t \leq 1$

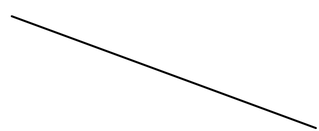
Graph of $_tp_x$



Uniform distribution of decrements

$s(x+t) = s(x)(1-t) + s(x+1)\cdot t \qquad 0 \leq t \leq 1$

Graph of $_tp_x$



Balducci (Hyperbolic) assumption ➔

$1/s(x+t) = (1-t) / s(x) + t / s(x+1) \qquad 0 \leq t \leq 1$

Graph of $_tp_x$

MORTALITY ASSUMPTIONS

Uniform distribution of decrements ➔ Linear interpolation of s(x) from age x to age x+1

$s(x+t)$	$= s(x)(1-t) + s(x+1) \cdot t$	for $0 \le t \le 1$
l_{x+t}	$= l_x(1-t) + l_{x+1}(t)$	for $0 \le t \le 1$
	$= l_x - t(d_x)$	

$$_tq_x = t(q_x)$$
$$\mu_{x+t} = q_x / (1-t(q_x))$$
$$_{1-t}q_{x+t} = (1-t)q_x / (1-t(q_x))$$
$$_yq_{x+t} = y(q_x) / (1-t(q_x))$$
$$_tp_x = 1 - t(q_x)$$
$$_tp_x\, \mu_{x+t} = q_x$$

Only valid for $0 \le t \le 1$
Formulas from Bowers Table 3.6.1

MORTALITY ASSUMPTIONS

Constant force of mortality ➔
Linear interpolation of $\ln$(s(x)) from age x to age x+1

$\ln[s(x+t)]$
$= \ln[s(x)]\cdot(1-t) + \ln[s(x+1)]\cdot t \quad$ **for $0 \le t \le 1$**

$\mu_{x+t} \quad = \mu \quad$ **for $0 \le t \le 1$**

${}_t p_x \quad = e^{-\int_x^{x+t} \mu_y \, dy} \quad = e^{-t\mu} \quad = (p_x)^t$

${}_t q_x \quad = 1 - (p_x)^t$

$\mu_{x+t} \quad = -\ln(p_x)$

${}_{1-t} q_{x+t} \quad = 1 - (p_x)^{1-t}$

${}_y q_{x+t} \quad = 1 - (p_x)^y$

${}_t p_x \, \mu_{x+t} \quad = -(p_x)^t \ln(p_x)$

Only valid for $0 \le t \le 1$
Formulas from Bowers Table 3.6.1

MORTALITY ASSUMPTIONS

Balducci (Hyperbolic) assumption → Harmonic interpolation of s(x) from age x to age x+1

$1/s(x+t) = (1-t) / s(x) + t / s(x+1) \qquad 0 \le t \le 1$

$1/l_{x+t} = (1-t) / l_x + t / l_{x+1} \qquad 0 \le t \le 1$

${}_tq_x = t(q_x) / [1-(1-t)q_x]$

$\mu_{x+t} = q_x / [1-(1-t)q_x]$

${}_{1-t}q_{x+t} = (1-t)q_x$

${}_yq_{x+t} = y(q_x) / [1-(1-y-t)q_x]$

${}_tp_x = p_x / [1-(1-t)q_x]$

${}_tp_x\,\mu_{x+t} = q_xp_x / [1-(1-t)q_x]^2$

Only valid for $0 \le t \le 1$

Formulas from Bowers Table 3.6.1

SELECT & ULTIMATE MORTALITY

Life is healthiest when insurance policy first issued (medical exam). After selection effect wears off, mortality only varies by age.

Issue Age	Select Durations				Ultimate Age
$[x]$	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
20	.0110	.0150	.0175	.0200	23
21	.0120	.0160	.0185	.0210	24
22	.0130	.0170	.0195	.0220	25
23	.0140	.0180	.0205	.0230	26

For policy issued at age 20, mortality rates are as follows:

$q_{[20]} = .0110 \qquad q_{23} = .0200$
$q_{[20]+1} = .0150 \qquad q_{24} = .0210$
$q_{[20]+2} = .0175 \qquad q_{25} = .0220$

CENTRAL DEATH RATE: m_x

$L_x = l_{x+1} + \int_0^1 t(l_{x+t})\mu_{x+t}\,dt$ Bowers 3.5.11

$= \int_0^1 l_{x+t}\,dt$ Bowers 3.5.12

= # years lived from x to x+1 <u>in a year</u> by those currently age x

$m_x = d_x / L_x$

Relates number of deaths (from x to x+1) to the mean value of l_x (from x to x+1)

CENTRAL DEATH RATE

$$m_x = \frac{\int_0^1 l_{x+t}\,\mu_{x+t}\,dt}{\int_0^1 l_{x+t}\,dt} = \frac{l_x - l_{x+1}}{L_x} \qquad \text{Bowers 3.5.13}$$

Weighted mean value of μ_x (from x to x+1) where weights are # lives attaining each age x+t in that interval

CENTRAL DEATH RATE

Assuming uniform distribution of decrements:

$L_x = l_x - ½(d_x)$

$m_x = q_x / [1 - ½(q_x)]$ (Bowers Exer. 3.45)
$q_x = m_x / [1 + ½(m_x)]$ (Bowers Exer. 3.45)

$p_x = [1 - ½(m_x)] / [1 + ½(m_x)]$

CENTRAL DEATH RATE

Multi-year definitions

$${}_nL_x = n l_{x+n} + \int_0^n t(l_{x+t})\mu_{x+t}\,dt$$

$$= \int_0^n l_{x+t}\,dt$$ **Bowers 3.5.14**

$${}_nm_x = \frac{\int_0^n l_{x+t}\mu_{x+t}\,dt}{\int_0^n l_{x+t}\,dt}$$ **Bowers 3.5.15**

$$= (l_x - l_{x+n}) / {}_nL_x$$

$$= {}_nd_x / {}_nL_x$$

Consistent with prior definition

$m_x = d_x / L_x$

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KEY CONCEPTS

SECTION VII - LIFE CONTINGENCIES

1. Mortality functions:
   l_x, $s(x)$, u_x

2. Probability definitions:
   p_x, q_x

3. Probability Ability

4. Mortality assumptions

5. Select and Ultimate

6. Central death rates