SECTION 10

JOINT LIFE STATUS

JOINT LIFE STATUS PROBABILITIES

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p_{xy} = Probability that two lives age x and y <u>both</u> live one year

$= p_x(p_y)$

$p_{\overline{xy}}$ = Probability that <u>at least one</u> of the two lives age x and y both live

$= p_x + p_y - p_{xy}$

$= 1 - (1-p_x)(1-p_y)$

JOINT LIFE STATUS ANNUITIES

$\ddot{a}_{xy}$ = Annuity payable while two lives age x and y are <u>both</u> alive

$$= 1 + vp_{xy} + v^2 {}_2p_{xy} + \ldots$$

$\ddot{a}_{\overline{xy}}$ = Annuity payable while <u>at least one</u> of the two lives age x and y are alive

$$= 1 + vp_{\overline{xy}} + v^2 {}_2p_{\overline{xy}} + \ldots$$

NOTE:
Formulas also work fine for monthly annuities.

JOINT LIFE STATUS ANNUITIES

Think in terms of simple probabilities

$p_{\overline{xy}} = p_x + p_y - p_{xy}$

Annuity formulas follow same pattern

$\ddot{a}_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}$

JOINT LIFE STATUS
THREE LIVES (OR MORE)

Use pattern from prior formulas to construct values for more than 2 lives

$$p_{\overline{xyz}} = p_x + p_y + p_z - p_{xy} - p_{xz} - p_{yz} + p_{xyz}$$

$$\ddot{a}_{\overline{xyz}} = \ddot{a}_x + \ddot{a}_y + \ddot{a}_z - \ddot{a}_{xy} - \ddot{a}_{xz} - \ddot{a}_{yz} + \ddot{a}_{xyz}$$

Not tested on EA-1 for MANY years

JOINT LIFE STATUS
THREE LIVES (OR MORE)

Use prior formulas to derive values for more than 2 lives

$$
\begin{aligned}
p_{\overline{xyz}} &= 1 - (1-p_x)(1-p_y)(1-p_z) \\
&= 1 - (1 - p_x - p_y + p_{xy})(1-p_z) \\
&= 1 - (1-p_x-p_y+p_{xy})+(p_z-p_{xz}-p_{yz}+p_{xyz}) \\
&= p_x + p_y + p_z - p_{xy} - p_{xz} - p_{yz} + p_{xyz}
\end{aligned}
$$

$$
\begin{aligned}
\ddot{a}_{\overline{yz}} &= \ddot{a}_y + (\ddot{a}_z - \ddot{a}_{yz}) \\
\ddot{a}_{\overline{xyz}} &= \ddot{a}_{\overline{x:yz}} \\
&= \ddot{a}_x + (\ddot{a}_{\overline{yz}} - \ddot{a}_{x:\overline{yz}}) \\
&= \ddot{a}_x + [\ddot{a}_y + (\ddot{a}_z - \ddot{a}_{yz})-(\ddot{a}_{xy} + \ddot{a}_{xz} - \ddot{a}_{xyz})] \\
&= \ddot{a}_x + \ddot{a}_y + \ddot{a}_z - \ddot{a}_{xy} - \ddot{a}_{xz} - \ddot{a}_{yz} + \ddot{a}_{xyz}
\end{aligned}
$$

JOINT LIFE STATUS
LAW OF UNIFORM SENIORITY

EXTRA CREDIT – not tested on exam
Gompertz' law
$\mu_x = Bc^x$

Using Gompertz' law above to construct mortality table, you can derive t such that $a_{x:x+n} = a_{x+t}$

Logically, you have x+t > x+n, since a_{x+t} annuity value must be smaller than a_{x+n}.

JOINT LIFE STATUS
LAW OF UNIFORM SENIORITY

EXTRA CREDIT – not tested on exam
Gompertz' law
$\mu_x = Bc^x$

You have x+t > x+n, so (x+t – x+n) is the addition to the older age.

Values for 1937 Standard Annuity table

Age difference	Add to older age
0	9.122
1	8.632
2	8.160

Values of t-n given in problems are always integer (unlike above)

JOINT LIFE STATUS
LAW OF UNIFORM SENIORITY

EXTRA CREDIT – not tested on exam
Makeham's law
$\mu_x = A + Bc^x$

Using Makeham's law above to construct mortality table, you can derive t such that

$$a_{x:x+n} = a_{x+t:x+t}$$

Logically, you have x+t < x+n, since you are replacing with two lives at age x+t.

JOINT LIFE STATUS
LAW OF UNIFORM SENIORITY

EXTRA CREDIT – not tested on exam
Makeham's law
$\mu_x = A+Bc^x$

You have x+t < x+n, so (x+t – x) is the addition to the younger age.

Values for 1941 CSO table

Age difference	Add to younger age
1	.511
2	1.043
3	1.596

Values of t given in problems are always integer (unlike above)

REVERSIONARY ANNUITY

This pays \$1 annually to life x after the death of life y:

$$\ddot{a}_{Y|X} = \ddot{a}_x - \ddot{a}_{xy}$$

As long as x and y are both alive, this annuity pays \$1 - \$1, or zero.

If x is alive and y is dead, the first annuity pays \$1, and the second pays 0.

REVERSIONARY ANNUITY

Pop Quiz

Three lives are ages x, y and z. Write an expression for the PV of an annuity that pays $500 per annum as long as exactly 2 of the three are alive.

ACTUARIAL EQUIVALENCE

If two annuities are actuarially equivalent, then they have the same present value:

Present value of normal form benefit
= Present value of optional form benefit

<u>EXTRA CREDIT – not on exam</u>
Must use unisex assumptions to determine present values

These optional form assumptions are typically NOT the same as those used to determine cost for pension plan

ACTUARIAL EQUIVALENCE

Problems may give you the annuity factors to determine actuarially equivalent benefits

Alternative is to give you the optional form reduction factor:

PV of normal form benefit of $1
= PV of optional form benefit of $B

$$B = \frac{\text{Normal form annuity factor}}{\text{Optional form annuity factor}}$$

B is the optional form reduction factor

JOINT AND SURVIVOR FACTOR

Formulas below are for a benefit that reduces <u>only</u> upon employee's death. This was defined in ERISA (1974) as a Joint and Survivor benefit ("J&S").

Assume life annuity of $1 that is actuarially equivalent to "J&S" annuity of $B, continues at rate K to beneficiary

PV of Normal Form		PV of Optional form
	=	

$$1 * (\ddot{a}_X) = B * (\ddot{a}_X + K(\ddot{a}_Y - \ddot{a}_{XY}))$$

$$B = \ddot{a}_X \; / \; (\ddot{a}_X + K(\ddot{a}_Y - \ddot{a}_{XY}))$$

$$B = 1 \; / \; \left(1 + K\frac{(\ddot{a}_Y - \ddot{a}_{XY})}{\ddot{a}_X}\right)$$

JOINT AND SURVIVOR

Formulas below are for a benefit that reduces only upon employee's death

If K = 50%

$$B = (1 + .50[(\ddot{a}_{Y} - \ddot{a}_{XY}) / \ddot{a}_{X}])^{-1}$$

If K = 100%

$$B = (1 + 1.0[(\ddot{a}_{Y} - \ddot{a}_{XY}) / \ddot{a}_{X}])^{-1}$$

EXTRA CREDIT

Given a "J&S" factor based on one continuation fraction (K), you can derive the "J&S" factor for any other value of K

Not tested on EA-1 exam for MANY years

JOINT AND SURVIVOR

Formulas on next page are for a benefit that reduces upon <u>either</u> the employee's death or joint annuitant's death. This is the correct definition of a Joint and Survivor benefit.

Due to confusion over "Joint and Survivor", problems will describe the benefit, and upon whose death the benefit reduces

JOINT AND SURVIVOR

Formulas below are for a benefit that reduces upon <u>either</u> the employee's death or joint annuitant's death.

Assume life annuity of $1 that is actuarially equivalent to <u>true</u> J&S annuity of $B, continues at rate K to survivor

PV of Normal Form	=	PV of Optional form

$$1*(\ddot{a}_X) = B*(K\ddot{a}_X + K\ddot{a}_Y + (1-2K)\ddot{a}_{XY})$$

$$B = \ddot{a}_X / (K\ddot{a}_X + K\ddot{a}_Y + (1-2K)\ddot{a}_{XY})$$

JOINT AND SURVIVOR

- **Can you derive the prior formula?**
  Hint - use reversionary annuity concept

- **This J&S factor B may be > 1.00**

- **Unlike other J&S annuity, can't solve for factors based on other values of K**

Comparison of J&S benefit types

For 100% continuation:

Reduce ee death $= \ddot{a}_X + K(\ddot{a}_Y - \ddot{a}_{XY})$

$= \ddot{a}_X + \ddot{a}_Y - \ddot{a}_{XY}$

Reduce either dies $= K\ddot{a}_X + K\ddot{a}_Y + (1-2K)\ddot{a}_{XY}$

$= \ddot{a}_X + \ddot{a}_Y - \ddot{a}_{XY}$

For 50% continuation:

Reduce ee death $= \ddot{a}_X + K(\ddot{a}_Y - \ddot{a}_{XY})$

$= \ddot{a}_X + .5(\ddot{a}_Y - \ddot{a}_{XY})$

Reduce either dies $= K\ddot{a}_X + K\ddot{a}_Y + (1-2K)\ddot{a}_{XY}$

$= .5\ddot{a}_X + .5\ddot{a}_Y$

Verify results for all three cases -
Both X+Y alive, only X alive, only Y alive

JOINT AND SURVIVOR WITH POP-UP

Benefit reduces only upon employee's death. If beneficiary dies first, then the employee's benefit is restored to life annuity amount, as if no J&S option was ever elected.

Construct pieces of J&S pop-up benefit:

$B\ddot{a}_X$	benefit for life of ee
$+ KB(\ddot{a}_Y - \ddot{a}_{XY})$	benefit if ee dies first
$+(1-B)(\ddot{a}_X - \ddot{a}_{XY})$	benefit if benef. dies first

JOINT AND SURVIVOR WITH POP-UP

PV of
Normal PV of
Form = Optional form

$$
\begin{aligned}
1(\ddot{a}_X) = {} & B\,\ddot{a}_X + KB\ddot{a}_Y - KB\ddot{a}_{XY} \\
& + \ddot{a}_X \qquad\qquad\quad - \ddot{a}_{XY} \\
& - B\,\ddot{a}_X \qquad\qquad\quad + B\ddot{a}_{XY}
\end{aligned}
$$

$$\ddot{a}_{XY} = KB\ddot{a}_Y + (1-K)B\ddot{a}_{XY}$$

$$B = \ddot{a}_{XY} / (K\ddot{a}_Y + (1-K)\ddot{a}_{XY})$$

Key result – final formula does not include $\ddot{a}_X$

JOINT AND SURVIVOR USEFUL FORMULAS

$$a_x = vp_x \ddot{a}_{x+1}$$

If given three of four items in prior formula, can solve for the unknown

$$a_{xy} = vp_{xy} \ddot{a}_{x+1:y+1}$$

If given three of four items in prior formula, can solve for the unknown

KEY CONCEPTS

SECTION X - JOINT LIFE STATUS

1. Joint Life Status
   a. Probability
   b. Annuity

2. Joint and Last Survivor
   a. Probability
   b. Annuity

3. Reversionary Annuity

4. Types of J&S Annuities