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SPRING 2015 EA-1
EXAM SOLUTIONS

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1 This problem may look familiar - it is identical to 2007 #11. I'll show a simpler method of solution than I used back in 2007. The key to the solution is knowing this identity formula:

$$\frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} = \frac{i^{(m)}}{m} * \frac{d^{(m)}}{m}$$

In this problem you have $\frac{i^{(12)}}{d^{(12)}} = 1.01 \Rightarrow i^{(12)} = 1.01(d^{(12)})$

Now you can substitute that in the formula above

$$\frac{1.01d^{(12)} - d^{(12)}}{12} = \frac{1.01d^{(12)} * d^{(12)}}{12*12}$$

$$\frac{d^{(12)}}{12}\left[1.01 - 1.0\right] = \frac{d^{(12)}}{12}\left[\frac{1.01d^{(12)}}{12}\right]$$

$$.01 = 1.01\frac{d^{(12)}}{12} \Rightarrow \frac{i^{(12)}}{12} = .01$$

The problem asks for the annual rate of interest

$$1+i = \left(1+\frac{i^{(12)}}{12}\right)^{12}$$

$$= (1+.01)^{12}$$

$$= 1.1268$$

D

2 This is a very straightforward question on using commutation functions. The problem asks for ${}_3q_{89}$, and gives you N_x values.

The key definition is $N_x - N_{x+1} = D_x$ and $D_x = v^x l_x$. You can express the probability in terms of l_x values, and use the N_x values to calculate D_x, and the value of the probability.

x	N_x	$D_x = N_x - N_{x+1} = v^x l_x$	
88	208	55	
89	153	43	$= v^{89} l_{89}$
90	110	33	
91	77	24	
92	53	18	$= v^{92} l_{92}$
93	35		

$${}_3q_{89} = 1 - {}_3p_{89} = 1 - l_{92}/l_{89}$$

$$\frac{D_{92}}{D_{89}} = \frac{v^{92} l_{92}}{v^{89} l_{89}} = v^3 \frac{l_{92}}{l_{89}} = \frac{18}{43}$$

$$\frac{l_{92}}{l_{89}} = (1.03)^3 (18/43) = .457$$

It is not correct to use more than 2 digits in this calculated value, since the N_x values only are given with two significant digits

$${}_3q_{89} = 1 - l_{92}/l_{89} = 1 - .46 = .54$$

E

3 This is a typical exam question on commutation functions. You must know several identities to work this problem:

$$N_x = N_{x+1} + D_x = \sum_{t=0}^{\omega} D_{x+t} \qquad a_x = \frac{N_{x+1}}{D_x}$$

$$_nE_x = \frac{D_{x+n}}{D_x} \qquad {}_{n|}a_x = \frac{N_{x+n+1}}{D_x} \qquad \ddot{s}_{x:\overline{n}|} = \frac{N_x - N_{x+n}}{D_{x+n}}$$

The problem asks for $\ddot{a}_x = \frac{N_x}{D_x}$. You are given values for $_{10|}a_x$ and $\ddot{s}_{x:\overline{10}|}$. The first item has payments starting after 10 years, and the second item has payments for only 10 years. By combining these, and valuing at age x, you can calculate the value of $\ddot{a}_x$.

$$_{10|}a_x = \frac{N_{x+11}}{D_x} = 7.0 \qquad _{10}E_x = \frac{D_{x+10}}{D_x} = .40$$

$$_{10|}a_x + {}_{10}E_x = \frac{N_{x+10}}{D_x} = 7.40$$

$$\ddot{s}_{x:\overline{10}|} = \frac{N_x - N_{x+10}}{D_{x+10}} = 15.0 \qquad \ddot{s}_{x:\overline{10}|}({}_{10}E_x) = \frac{N_x - N_{x+10}}{D_x} = 15.0(.40) = 6.0$$

$$_{10|}a_x + {}_{10}E_x = \frac{N_{x+10}}{D_x} = 7.40$$

$$\Sigma = \frac{N_x}{D_x} = \ddot{a}_x = 13.40$$

E

4 This problem may look familiar - it is identical to 2007-21. I'll use a slightly different solution technique to work it this time.

The first step is writing down the series of payments on a time line diagram

You can write out a formula for X fairly easily:

$$X = 25{,}000\left(v^1 + 2v^2 + 3v^3 + \ldots + 10v^{10} + 10v^{11} + \ldots\right)$$

This is the sum of an increasing annuity and a perpetuity. That idea was used in the 2007 solution, so this time I'll use a different approach. I will multiply both sides by v^1, then subtract the equations

$$X \cdot v^1 = 25{,}000\left(\quad v^2 + 2v^3 + \ldots + 9v^{10} + 10v^{11} + \ldots\right)$$
$$X(1-v^1) = 25{,}000\left(v^1 + v^2 + v^3 + \ldots + v^{10} + 0 + 0 + \ldots\right)$$

$$X = 25{,}000\left(\frac{a_{\overline{10}|.05}}{1-v}\right)$$
$$= 25{,}000\left(\frac{7.7217}{.0476}\right)$$
$$= 4{,}053{,}911$$

B

5 This is a typical exam question on calculating duration for a bond. It is similar to 2009-09, which was for a 5 year bond.

This problem does not tell you the yield rate. Based on the statement that the bond is "purchased and redeemed at par", you know that the yield rate must be the same as the 5.5% annual coupon rate.

You must know the formula for duration to work this!

Duration $\bar{d} = \dfrac{\sum t v^t R_t}{\sum v^t R_t}$ where R_t is the cash flow, dividends/coupons/redemption

$$\bar{d} = X = \frac{F\left[.055(1v^1 + 2v^2 + \ldots + 25v^{25}) + 25v^{25}\right]}{F\left[.055(v^1 + v^2 + \ldots + v^{25}) + v^{25}\right]}$$

F is the face amount of the bond, which cancels out. To simplify this calculation, you must know the formula for an increasing annuity:

$$X = \frac{.055(Ia_{\overline{25}|.055}) + 25v^{25}}{.055(a_{\overline{25}|.055}) + v^{25}} \quad \Rightarrow \quad i(a_{\overline{n}|i}) + v^n = 1.0$$

$$Ia_{\overline{n}|i} = (\ddot{a}_{\overline{n}|} - nv^n)/i = \left[(1+i)a_{\overline{n}|i} - nv^n\right]/i$$

$$Ia_{\overline{25}|.055} = 1.055(a_{\overline{25}|.055} - 25v^{25})/.055 = 138.1065$$

$$X = \frac{(138.1065)(.055) + 6.5555}{1.0} = 14.152 \qquad \text{B}$$

6 This is a very simple question on the definition of a mortality table setback. You should set up the probability definition ignoring the setback, then adjust the ages back two years:

ignoring setback $X = {}_7p_{58}$

$$= \frac{l_{65}}{l_{58}}$$

2 year setback $X = \frac{l_{63}}{l_{56}}$

$$= \frac{7,195,099}{8,223,010}$$

$$= .875$$

E

7 This is a typical question on multiple decrement tables. The key to working the problem is using the correct formulas based on uniform distribution of decrements (UDD) in the single decrement tables:

UDD-single decrement table

$$q'^{(i)}_x \doteq \frac{q^{(i)}_x}{1-\frac{1}{2}q'^{(2)}_x}$$

UDD-multiple decrement table

$$q'^{(i)}_x \doteq \frac{q^{(i)}_x}{1-\frac{1}{2}q^{(2)}_x}$$

$$q'^{(w)}_x \doteq \frac{q^{(w)}_x}{1-\frac{1}{2}q'^{(d)}_x} \Rightarrow X \doteq \frac{5q^{(d)}_{50}}{1-.50(.035)}$$

$$q'^{(d)}_x \doteq \frac{q^{(d)}_x}{1-\frac{1}{2}q'^{(w)}_x} \Rightarrow .035 \doteq \frac{q^{(d)}_{50}}{1-.50(X)}$$

$$q^{(d)}_{50} \doteq .035[1-.5X]$$

Now substitute the formula for $q^{(d)}_{50}$ into the formula for X, and you can calculate X:

$$X \doteq \frac{5(.035)[1-.5X]}{1-.175}$$

$$.5825X \doteq .1750-.0875X$$

$$X \doteq .1636$$

B

8 This is a typical exam question on actuarially equivalent joint and survivor annuity benefits. This problem is quite similar to 2003-24. When two forms of payment are actuarially equivalent, their present values are equal.

In this problem you are not given any annuity factors. The key to working the problem is writing down the formulas correctly for the J+S benefits. Once you do that, it becomes an algebra problem – "can you solve for X."

PV1: $15{,}000\,\ddot{a}_z^{(12)}$

Since X is the unknown, I will use z for Smith's age, and y for Smith's spouse's age.

PV2: $15{,}000\,\ddot{a}_z^{(12)} = 13{,}500\,\ddot{a}_z^{(12)} + 6{,}750\left(\ddot{a}_y^{(12)} - \ddot{a}_{yz}^{(12)}\right)$

PV3: $15{,}000\,\ddot{a}_z^{(12)} = X\,\ddot{a}_z^{(12)} + .75X\left(\ddot{a}_y^{(12)} - \ddot{a}_{yz}^{(12)}\right)$

The trick to the problem is getting rid of all the annuity factors. Otherwise, you have too many unknowns to solve! Divide both sides by $\ddot{a}_z^{(12)}$:

PV2: $15{,}000 = 13{,}500 + 6{,}750\left[\left(\ddot{a}_y^{(12)} - \ddot{a}_{yz}^{(12)}\right)/\ddot{a}_z^{(12)}\right]$

PV3: $15{,}000 = X + .75X\left[\left(\ddot{a}_y^{(12)} - \ddot{a}_{yz}^{(12)}\right)/\ddot{a}_z^{(12)}\right]$

(8) continued

Now you can replace the ratio of the annuities by R - and you can calculate the values of X and R easily.

PV2: $15{,}000 = 13{,}500 + 6{,}750\,R$

$R = .222$

PV3: $15{,}000 = X + .75X(R)$

$$X = \frac{15{,}000}{[1 + .75(.222)]}$$

$= 12{,}857$

D

9 This is a very simple question on asset values, realized and unrealized gains and losses. A realized gain occurs when you sell an asset, and it affects both the book value and market value of the assets.

An unrealized gain is the excess of the market value over the book value of the assets. A reconciliation of the market value of assets from one year to the next includes the realized gain/loss as well as the change in the unrealized gain/loss.

This problem asks for the unrealized gain during 2014, but that terminology is incorrect. It really is the change in the unrealized gain.

You are given the account balance at 01/2014, which is probably the market value. You are given the dollar weighted return, so you can also calculate the market value at the end of 2014. Then you can derive the value of X:

$$12/31/14 \text{ MVA} = 500{,}000(1.1846) + 100{,}000\left(1 + \frac{5}{12}(.1846)\right)$$
$$= 699{,}992$$

$$699{,}992 = 500{,}000 + 100{,}000 \text{ deposit} + 20{,}000 \text{ RG} + X \text{ unrealized gains}$$

$$X = 69{,}992$$

A

10 This is a typical exam question on accumulated fund values. This is very similar to 2014-20, and other recent exam questions. Smith deposits 2% of pay at the end of each year, and their pay goes up 4% each year.

The first step is writing down all the fund deposits on a time line diagram. Then you can write a formula for the accumulated value at 01/01/2030:

$$X = 1{,}000\left[(1.05)^{14} + (1.04)^1(1.05)^{13} + \ldots + (1.04)^{13}(1.05)^1 + (1.04)^{14}\right]$$

Note the sum of exponents is 14 in each term. The standard simplification with a series like this is to factor out the first term (or the last term):

$$X = 1{,}000(1.05)^{14}\left[1 + \left(\frac{1.04}{1.05}\right)^1 + \ldots + \left(\frac{1.04}{1.05}\right)^{13} + \left(\frac{1.04}{1.05}\right)^{14}\right]$$

$$= 1{,}000(1.05)^{14}\,\ddot{a}_{\overline{15}|j} \quad \text{where } 1+j = \frac{1.05}{1.04} \Rightarrow j = .96\%$$

$$= 1{,}000(1.9799)(1.0096)\,a_{\overline{15}|.0096}$$

$$= 27{,}998$$

B

11. This is a typical question on De Moivre's law, which is defined by $l_x = w - x$. The problem tells you deaths are uniformly distributed, which means the same number of deaths at each age. You can define the value of l_0 as anything you want, but the simplest approach is to use $l_x = 100 - x$.

The question asks for the value of $100a_{60}$. You could write a formula and derive the value, but this is very slow. A much better approach (since this is tested routinely) is to memorize the formula for a_x:

$$a_x = \frac{n - \ddot{a}_{\overline{n}|i}}{ni} \quad \text{where } n = w - x$$

$$a_{60} = \frac{40 - \ddot{a}_{\overline{40}|.06}}{40(.06)} \qquad n = 100 - 60 = 40$$

$$= \frac{40 - 1.06(15.0463)}{40(.06)}$$

$$= 1.002$$

B

Note that I use the HP-12C calculator, and always calculate end of year annuities for EA-1 exam problems. This avoids silly arithmetic / logic errors that could occur if I mistakenly calculate an annuity due instead of an annuity immediate.

12 This may look familiar to you - this concept is tested frequently on the exam, but this problem is identical to 2012-11. The key point is whether you know the definition of the expectation of life:

$$e_x = {}_1p_x + {}_2p_x + {}_3p_x + \ldots$$

The problem gives values of e_x at successive ages, and asks for the number of survivors from age 75 to 77. You can set up a formula for the probability of survival:

$$e_x = {}_1p_x(1 + e_{x+1})$$

$${}_1p_x = e_x / (1 + e_{x+1})$$

$$p_{75} = e_{75}/(1+e_{76}) = 10.5/(1+10.0) = .9545$$

$$p_{76} = e_{76}/(1+e_{77}) = 10.0/(1+9.6) = .9434$$

$${}_2p_{75} = p_{75}(p_{76}) = .9545(.9434) = .9005$$

$$10{,}000({}_2p_{75}) = 9{,}005$$

B

13 This is a typical question on probability calculations involving mortality tables. This may seem familiar, since it is identical to the 2007 exam problem 9!

The question asks for the value of $\ddot{a}_{60:62:\overline{3}|}$. This is a joint life annuity, but it only has three payments. You need to write the value of the annuity in terms of p_x and interest:

$$\ddot{a}_{60:62:\overline{3}|} = 1 + v\,p_{60}p_{62} + v^2\,{}_2p_{60}({}_2p_{62})$$

$$= 1 + (1.05)^{-1}p_{60}p_{62}\left[1 + (1.05)^{-1}(p_{61})(p_{63})\right]$$

$$= 1 + \frac{(1-.0156)(1-.0184)}{1.05}\left[1 + \frac{(1-.0169)(1-.0201)}{1.05}\right]$$

$$= 1 + .9844(.9816)(.9524)\left[1 + .9831(.9799)(.9524)\right]$$

$$= 2.765$$

E

14 This is a typical loan problem, but it does have a change in the amount of payment and the remaining term. The first step is to write down the payments on a time line diagram. I'll use P to represent the original loan payment

You are told that the interest rate is 12% per annum, compounded monthly. You can calculate the value of P based on the monthly interest rate of 1%:

$$100{,}000 = P\, a_{\overline{360}|1\%}$$
$$P = 1{,}028.61$$

The loan is renegotiated after 120 payments are made. At that time, the outstanding loan balance is the present value of the remaining 240 payments

$$\text{O/S balance} = P\, a_{\overline{240}|1\%}$$
$$= 100{,}000 \left(\frac{a_{\overline{240}|i}}{a_{\overline{360}|i}} \right)$$
$$= 93{,}418$$

Now the loan payment increases to P + X, based on only making 40 more loan payments.

(14) continued

$$93{,}418 = (P + X)\, a_{\overline{40}|1\%}$$

$$P + X = \frac{93{,}418}{a_{\overline{40}|1\%}}$$

$$X = \frac{93{,}418}{32.8347} - 1028.61$$

$$\approx 1{,}816.49$$

D

This problem is fairly similar to 2012 exam #5.

15 Questions on bonds rarely involve the idea of "premium" versus "discount". This refers to the difference between the price paid for a bond and the redemption value of the bond. This problem may seem familiar to you - it is identical to 2005 exam problem 21.

You can calculate the price of the bond using any of the formulas. Just to differ from the 2005 solution, I'll use the standard bond price formula:

$$P = Fr(a_{\overline{n}|i}) + Cv^n$$
$$= 1000\left(\frac{.05}{2}\right)a_{\overline{20}|i} + 1000v^{20}$$

One key to the problem is that the coupon rate is semi-annual, but the yield rates are given as annual. You need to convert those rates to semi-annual rates:

4% annual Yield	6% annual yield
$i = (1.04)^{\frac{1}{2}} - 1 = 1.98\%$	$i = (1.06)^{\frac{1}{2}} - 1 = 2.96\%$
$P = 1000(.025)a_{\overline{20}|1.98\%} + 1000(1.0198)^{-20}$	$P = 1000(.025)a_{\overline{20}|2.96\%} + 1000(1.0296)^{-20}$
$= 25(16.38) + 1000(.6756)$	$= 25(14.94) + 1000(.5584)$
$= 1085.12$	$= 931.84$
$X = P - 1000 = 85.12$	$Y = 1000 - P = 68.16$

$$|X| - |Y| = 16.96 \quad \text{E}$$

16 This is a typical question on actuarially equivalent benefits. If two benefits are actuarially equivalent, then they have the same present values.

The first step in this problem is writing down the formulas for each present value:

Normal form $PV = 1500\,\ddot{a}_{65}^{(12)}$

optional form $PV = X(v^5\,{}_5p_{65})\left[\ddot{a}^{(12)}_{\overline{5}|66} + v^5\,{}_5p_{70}\,\ddot{a}^{(12)}_{75}\right]$

You are given the N_x and D_x commutation functions, so you should calculate the values for $\ddot{a}_{65}^{(12)}$ and $\ddot{a}_{75}^{(12)}$ as the next step. Then you can calculate X:

$$\ddot{a}_{65}^{(12)} = \ddot{a}_{65} - 11/24 = \frac{9900}{1000} - .4583 = 9.4417$$

$$\ddot{a}_{75}^{(12)} = \ddot{a}_{75} - 11/24 = \frac{3{,}160}{400} - .4583 = 7.4417$$

$$1500\,\ddot{a}_{65}^{(12)} = X\,(v^5\,{}_5p_{65})\left[\text{(as above)}\right]$$

$$X = \frac{1500\,\ddot{a}_{65}^{(12)}}{v^5\,{}_5p_{65}\left[\ddot{a}^{(12)}_{\overline{5}|66} + v^5\,{}_5p_{70}\,\ddot{a}^{(12)}_{75}\right]}$$

$$= \frac{1500(9.4417)}{\frac{D_{70}}{D_{65}}\left(\ddot{a}^{(12)}_{\overline{5}|66}\right) + \frac{D_{75}}{D_{65}}(7.4417)}$$

(16) continued

There are two ways to calculate the value for $\ddot{a}^{(12)}_{\overline{5}|6\%}$. One involves adjusting the annual annuity, the other involves using a monthly rate to calculate the annuity.

I prefer to convert the 6% annual rate to a monthly interest rate:

$$(1+j)^{12} = 6\% \Rightarrow j = (1.06)^{\frac{1}{12}} - 1$$
$$= .4868\%$$

$$\ddot{a}^{(12)}_{\overline{5}|6\%} = \frac{1}{12}\left(\ddot{a}_{\overline{60}|.4868\%}\right)$$
$$= \left(\tfrac{1}{12}\right)(1.004868)(51.9238) \Rightarrow \text{calculated } (1+j)\,a_{\overline{60}|j}$$
$$= 4.3480$$

Now you can finish plugging in the values to calculate the value of X:

$$X = \frac{1500(9.4417)}{\frac{D_{70}}{D_{65}}(4.3480) + \frac{D_{75}}{D_{65}}(7.4417)}$$
$$= \frac{14{,}162.50}{(730(4.3480) + 400(7.4417))/1000}$$
$$= 2{,}303 \qquad \text{B}$$

17 This is a fairly short and sweet question on life insurance. You are given an annuity value, and you need to calculate a life insurance value. This problem may seem familiar to you - it is identical to 2012 exam problem 21.

The key to working the problem is knowing the relationship between annuities and insurances:

$$A_x = 1 - d\ddot{a}_x$$

You can calculate the discount rate d based on 6% interest:

$$d = iv = .06/1.06$$
$$= .0566$$

Now you must calculate $\ddot{a}_{60}$ based on the probability of survival and the age 61 immediate annuity. The key is knowing the relationship for annuity values at successive ages:

$$a_x = v p_x \ddot{a}_{x+1}$$
$$\ddot{a}_x = 1 + v p_x (1 + a_{x+1})$$
$$\ddot{a}_{60} = 1 + (1.06)^{-1}(1 - q_{60})(1 + a_{61})$$
$$= 1 + .9434(1 - .004803)(12.5069)$$
$$= 12.7423$$

(17) continued

The question asks for the net single premium for the insurance at age 60:

$$X = 1000 A_{60}$$
$$= 1000 (1 - d\ddot{a}_{60})$$
$$= 1000 (1 - .0566(12.7423))$$
$$= 278.74$$

B

18 This is a typical exam question involving spot rates and the yield curve. The problem asks for the change in market value of the three bonds from the beginning to the end of the year. The key to working the problem is recognizing which rates to use at the begining of the year versus the end of the year.

Since you have zero coupon bonds, the market value is the discounted redemption value. It should be clear how you calculate the market value at the begining of the year:

MV bond 1 = $10{,}000(1.0275)^{-3}$ = 9,218
MV bond 2 = $20{,}000(1.0325)^{-5}$ = 17,044
MV bond 3 = $30{,}000(1.0350)^{-7}$ = 23,580
49,842 at B.O.Y.

When you value the bonds at the end of the year, they all are 1 year closer to the maturity date. Bond 1 will be redeemed in 2 years, bond 2 in 4 year, and bond 3 will be redeemed in 6 years:

MV bond 1 = $10{,}000(1.0075)^{-2}$ = 9,852
MV bond 2 = $20{,}000(1.0125)^{-4}$ = 19,030
MV bond 3 = $30{,}000(1.0175)^{-6}$ = 27,034
55,916 at E.O.Y.

X = 55,916 − 49,842
= 6,074

E

19 This is a typical exam question on select and ultimate mortality tables. The relationship between the select and ultimate q_x values is somewhat unusual.

Based on the formula for l_x, you can determine the values of l_x and q_x for ages 68 through 71:

$l_x = 1000 - 10x$
$q_x = 10/l_x$

Age	l_x	q_x
68	$1000-680=320$	$10/320$
69	$1000-690=310$	$10/310$
70	300	$10/300$
71	290	$10/290$

The key to the problem is correctly writing a formula for the value of $l_{[68]+2} = l_{[68]}(p_{[68]})(p_{[68]+1})$
$= l_{[68]}[1-q_{[68]}][1-q_{[68]+1}]$

You don't know the value of $l_{[68]}$, so you really can't get to the value of $l_{[68]+2}$ this way! Instead, you need to think about approaching it from the other direction — using the value of $l_{[68]+3} = l_{71}$:

$l_{[68]+3} = l_{[68]+2}(p_{[68]+2})$
$290 = l_{[68]+2}(1-.90q_{68+2})$
$l_{[68]+2} = 290/[1-.90(10/300)] = 298.97$ D

20 This is a somewhat interesting question on joint life annuity values. You are not given any annuity values, but you instead have present values of various annuities.

Since you do not have annuity values, I am going to simplify the solution by ignoring the monthly payments for the annuities. You are not given any ages either, so I'll use S for Smith's age and J for Jones' age.

Smith alive $a_S = 100$
Jones alive $a_J = X$
Smith only alive $a_S - a_{SJ} = 20$
Jones only alive $a_J - a_{SJ}$
OR Smith only alive + $a_S - a_{SJ} = 50$

The key to writing down the last two present values is using reversionary annuity formulas. The difference between a_S and a_{SJ} only provides a payment to Smith after the death of Jones. If both Smith and Jones are alive, then the net payment is zero.

$$a_S - a_{SJ} = 20 \Rightarrow a_{SJ} = 80$$

$$a_J - a_{SJ} + a_S - a_{SJ} = 50$$
$$X - 80 + 20 = 50$$
$$X = 110$$

D

21 This question asks about the central death rate m_x. These questions are not common on the exam, and you must know several formulas. This problem is identical to 2004 exam problem 11.

$m_x = d_x / L_x$ is the basic definition

$$= \frac{\int_0^1 l_{x+t}\, \mu_{x+t}\, dt}{\int_0^1 l_{x+t}\, dt}$$

If you assume uniform distribution of decrements, you have several more useful formulas:

$$L_x = l_x - \tfrac{1}{2}(d_x)$$
$$m_x = q_x / [1 - \tfrac{1}{2}(q_x)]$$
$$q_x = m_x / [1 + \tfrac{1}{2}(m_x)]$$
$$p_x = [1 - \tfrac{1}{2}(m_x)] / [1 + \tfrac{1}{2}(m_x)]$$

We'll use the first two formulas to work this problem:

$$L_x = l_x - \tfrac{1}{2}(d_x)$$
$$l_x - L_x = \tfrac{1}{2} d_x$$
$$2(l_x - L_x) = d_x$$
$$2(d_x + l_{x+1} - L_x) = d_x$$
$$2(d_x - 15) = d_x$$
$$d_x = 30$$
$$l_x = 990$$

$$m_x = q_x / [1 - \tfrac{1}{2}(q_x)]$$
$$= \frac{30/990}{1 - \tfrac{1}{2}(30/990)}$$
$$= .0308$$

$1000 m_x = 30.77$ (C)

22 This is a very messy loan problem - definitely seems more than a 3 point question to me. This problem is very similar to 2006 exam question 5.

The loan payments are NOT level, but the principal portion of each payment increases. The problem asks for the <u>present value</u> of the interest portion of the ten loan payments. You must construct the loan amortization table to set up the PV calculation:

Pmt #	Interest	Principal	O/S Loan
0			5,500

orig loan $= 100(1+2+\ldots+9+10) = 100(10)(11)/2 = 5{,}500$

1	$.04(5{,}500)$	$100(1)$	$5{,}500-100(1)$
2	$.04(5{,}500-100(1))$	$100(2)$	$5{,}500-100(1+2)$
3	$.04(5{,}500-100(1+2))$	$100(3)$	$5{,}500-100(1+2+3)$
$\vdots$		$\vdots$	$\vdots$
10	$.04(5{,}500-100(1+2+\ldots+9))$	$100(10)$	$5{,}500-100(1+2+\ldots+10)=0$

To calculate the present value of the interest payments, you should separate out the increasing amounts

$$X = .04(5{,}500)(v^1+v^2+\ldots+v^{10})$$
$$-.04(100)(1v^2+3v^3+\ldots+45v^{10})$$

(22) continued

To simplify this expression takes a bit more work. But you need to show a few more terms of this equation first

$$X = .04\left[5{,}500\, a_{\overline{10}|.04} - 100\left(1v^2 + 3v^3 + 6v^4 + \ldots + 36v^9 + 45v^{10}\right)\right.$$

$$v^1 X = .04\left[5{,}500\, v(a_{\overline{10}|}) - 100\left(1v^3 + 3v^4 + \ldots + 28v^9 + 36v^{10} + 45v^{11}\right)\right.$$

(now subtract these two equations)

$$X(1-v^1) = .04\left[5500(1-v)a_{\overline{10}|} - 100\left(1v^2 + 2v^3 + 3v^4 + \ldots + 9v^{10} - 45v^{11}\right)\right]$$

Now I'll substitute: $d = 1 - v = iv$ $\quad a_{\overline{10}|} = (1-v^{10})/i$

$$X(iv) = .04\left[5500\, v(1-v^{10}) - 100v\left(Ia_{\overline{9}|} - 45v^{10}\right)\right]$$

$$= .04v\left[5500 - 5500v^{10} - 100\, Ia_{\overline{9}|} + 4{,}500v^{10}\right]$$

$$.04X = .04\left[5500 \qquad -100\, Ia_{\overline{10}|.04}\right]$$

The last key to the solution is knowing the formula for $Ia_{\overline{n}|} = \dfrac{\ddot{a}_{\overline{n}|} - nv^n}{i}$

$$X = 5{,}500 - 100\left(\frac{\ddot{a}_{\overline{10}|} - 10v^{10}}{.04}\right)$$

$$= 5{,}500 - 2500\left(1.04(a_{\overline{10}|.04}) - 10(1.04)^{-10}\right)$$

$$= 5{,}500 - 2500(8.4353 - 6.7556)$$

$$= 1{,}301$$

D

23 This is <u>almost</u> a typical exam question on dollar weighted returns. One subtle trick is that the data covers both 2015 and 2016. If you don't notice that, you'll get a return that is at least twice as large as the correct answer.

The other item is the rather unusual wording: "X = the annualized rate of return from 1/1/2015 to 10/1/2016." It is also unusual to calculate an annual rate of return for number of months other than 12 or 24.

The first step is writing down the fund values and cash flows on a time line diagram:

I'll represent the annual rate of return by j:

$$50{,}000\left[1+\tfrac{21}{12}j\right]+4{,}000\left[1+\tfrac{18}{12}j\right]-7{,}000\left[1+\tfrac{3}{12}j\right]=52{,}000$$

$$j\left[50{,}000\left(\tfrac{21}{12}\right)+4{,}000\left(\tfrac{18}{12}\right)-7{,}000\left(\tfrac{3}{12}\right)\right]=5{,}000$$

$$j=5{,}000\,/\,[87{,}500+6{,}000-1{,}750]$$

$$=5.45\%$$

A

Just for fun, I tried using iteration to solve for j. After about 14 iterations, the value was closer to 5.31% than 5.45%.

24. The idea of projected mortality improvements has been tested routinely on recent exams. This question is similar to 2013 exam problem 24 – but that was a bit longer to work. The correct interpretation of the "3% mortality decrease" is that the projected q_x value after n years is equal to $q_x(1-.03)^n$.

An annuity is purchased at 01/01/2024 at age 50. You need to project the future mortality rates at ages 50 and 51, since this is an annuity due with only 3 payments:

	1-1-2014	1-1-2024	1-1-2025
Age x	q_x	q_x	q_x
50	.0157	$.0157(1-.03)^{10} = .01158$	
51	.0167		$.0167(1-.03)^{11} = .01195$
52	.0176		
53	.0187		

$$X = 10{,}000(1 + v p_{50} + v^2 {}_2p_{50})$$
$$= 10{,}000(1 + (1.05)^{-1}(1-.01158) + (1.05)^{-2}(1-.01158)(1-.01195))$$
$$= 10{,}000(1 + .941 + .886)$$
$$= 28{,}270$$

B

I intentionally rounded the factors to only 3 digits, since the q_x values given only have 3 significant digits.

25 This is a typical question on commutation functions, and basic identity formulas. You are given D_{49} and the question asks for N_{50}. The basic idea is to use the life insurance A_{50} to determine the life annuity $\ddot{a}_{50} = \frac{N_{50}}{D_{50}}$.

$D_x = v^x l_x$

$D_{49} = v^{49} l_{49} \Rightarrow l_{49} = (1.05)^{49}(6587.11)$
$= 71{,}940$

$l_{50} = l_{49} - d_{49}$
$= 71{,}940 - 790$
$= 71{,}150$

$D_{50} = v^{50} l_{50}$
$= (1.05)^{-50}(71{,}150)$
$= 6{,}204.55$

$A_{50} = 1 - d\ddot{a}_{50}$

$\ddot{a}_{50} = \frac{1 - A_{50}}{d} = \frac{1 - A_{50}}{iv} = \frac{N_{50}}{D_{50}} \Rightarrow N_{50} = \frac{D_{50}(1 - A_{50})}{.05/1.05}$

$= \frac{6204.55(1 - .450426)}{.04762}$

$= 71{,}607$

B

26 This is a typical exam question on calculating the duration for a perpetuity. If you knew the result from 2010 exam question 8, this could be a VERY short problem!

Here is the formula for the duration of a portfolio. The numerator is the weighted time to each future cash flow, and the denominator is the present value of the cash flows. The weighting in the numerator is the present value of each cash flow:

$$\text{Duration } \bar{d} = \frac{\sum t\, v^t R_t}{\sum v^t R_t}$$

where R_t is the cash flow, t is the time of payment

$$X = \frac{25{,}000[1 \cdot v^1 + 2v^2 + 3v^3 + \dots]}{25{,}000[v^1 + v^2 + v^3 + \dots]}$$

The denominator is the present value of a perpetuity immediate, which is $1/i$. The numerator is the present value of an increasing perpetuity immediate:

$$\begin{aligned} Y &= 1 \cdot v^1 + 2v^2 + 3v^3 + \dots \\ vY &= \phantom{1 \cdot v^1 +{}} 1v^2 + 2v^3 + \dots \\ (1-v)Y &= 1 \cdot v^1 + 1v^2 + 1v^3 + \dots \\ &= 1/i \\ \therefore Y &= [(1-v)i]^{-1} \end{aligned}$$

(26) continued

$$\bar{d} = \frac{[(1-v)i]^{-1}}{1/i}$$

$$= \frac{i}{(1-v)i}$$

$$= \frac{1}{1-v}$$

$$= \frac{1}{d}$$ which is present value of perpetuity due

$$= \frac{1}{1-(1.10)^{-1}}$$

$$= 11.0$$

E

The result from 2010 exam question 8 was that the <u>modified</u> duration of a perpetuity immediate is the same as the present value, or $1/i$.

$$\bar{d} = \text{regular duration} = \frac{1}{d} = \frac{1}{iv} = \frac{1+i}{i}$$

$$\frac{\bar{d}}{1+i} = \underline{\text{modified}} \text{ duration} = \frac{1}{i} = \left(\frac{1+i}{i}\right)\left(\frac{1}{1+i}\right)$$

27 This is a typical exam question on yield curves, spot rates and forward rates. The problem gives you the spot rates at the beginning of the year. These represent the yield to maturity for cash flows at $t=1$ and $t=2$. Most people think of the spot rates as "the yield curve".

The problem gives you the purchase price of a three year bond at $t=0$. Using this, you can derive the spot rate for $t=3$. But this can get too confusing for the solution, which requires you to calculate the sale price of the bond at the end of the year – and the yield curve is now different!

I think it makes more sense to solve for the yearly forward interest rates, since those rates are implicit in both yield curves ($t=0$ and $t=1$), and the forward rates do not change.

The problem asks for the "total realized return" on the bond. This wording is not 100% clear, but my best interpretation is that X must be the sum of the \$40 coupon received at $t=1$, and the difference between the sale price at $t=1$ and the purchase price of 974.21.

(27) continued

I won't use spot rates for the solution, but this is how you would solve for the year 3 rate:

$$974.21 = 40(1.03)^{-1} + 40(1.04)^{-2} + 1040(1+s)^{-3}$$

I will instead use the forward interest rates. The first step is deriving the 2nd year forward rate:

$$(1.04)^{-2} = (1.03)^{-1}(1+k_2)^{-1}$$
$$(1.04)^2/1.03 = (1+k_2) \Rightarrow k_2 = 5.01\%$$

Now you can use the price of the bond to derive the 3rd year forward rate:

$$974.21 = 40(1.03)^{-1} + 40(1.03)^{-1}(1.0501)^{-1} + 1040(1.03)^{-1}(1.0501)^{-1}(1+k_3)^{-1}$$
$$= 38.83 + 36.98 + 961.54(1+k_3)^{-1}$$
$$(974.21 - 38.83 - 36.98)/961.54 = (1+k_3)^{-1} \Rightarrow k_3 = 7.03\%$$

Now you can calculate the sale price of the bond at $t=1$, based on the 2nd and 3rd year forward rates:

$$P = 40(1.0501)^{-1} + 1040(1.0501)^{-1}(1.0703)^{-1}$$
$$= 963.44$$

$$X = 40 \text{ coupon} + 963.44 \text{ sale price} - 974.21 \text{ purchase price}$$
$$= 29.23$$

D

28 This is a typical exam question on multiple decrement tables. This may look familiar to you, since it is identical to 2012 exam #24.

You have information on exits for four different decrements in a multiple decrement table. You are told that there is uniform distribution of decrements in the multiple decrement table. The question asks for the rate of mortality in the single decrement table.

Based on U.D.D. in the multiple decrement table you can use this formula to directly calculate the rate of mortality:

$$ {}_tp_x'^{(1)} = \left[{}_tp_x^{(\tau)}\right]^{q_x^{(1)}/q_x^{(\tau)}} $$

also true for constant force of decrements in multiple table

$$ X = q_{40}'^{(d)} = 1 - p_{40}'^{(d)} $$

$$ p_{40}'^{(d)} = \left[p_{40}^{(\tau)}\right]^{q_{40}^{(d)}/(1-p_{40}^{(\tau)})} $$

$$ q_{40}^{(d)} = 12/1500 = .0080 \qquad q_{40}^{(\tau)} = (30+10+20+12)/1500 = .0480 $$

$$ p_{40}'^{(d)} = (1-.0480)^{.008/.048} $$
$$ = .9918 $$

$$ q_{40}'^{(d)} = 1 - .991835 $$
$$ = .008165 $$

B

The 2012 exam solution used a different formula (an approximation).

29. This is a typical exam question on select and ultimate decrements. This may look familiar – this is identical to 2011 exam problem 12.

The problem asks for ${}_4q_{[71]+1} = 1 - {}_4p_{[71]+1}$. This can be calculated using these q_x values from the table in the problem:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
71		.070	.082	73
72			.085	74
73			.088	75

Note that the heading shown in the problem is incorrect. In general, the right hand column of decrements is no longer select, so I have labelled it as q_{x+2}, not $q_{[x]+2}$.

$$
\begin{aligned}
X = {}_4q_{[71]+1} &= 1 - {}_4p_{[71]+1} \\
&= 1 - p_{[71]+1}(p_{73})(p_{74})(p_{75}) \\
&= 1 - (1-q_{[71]+1})(1-q_{73})(1-q_{74})(1-q_{75}) \\
&= 1 - (1-.070)(1-.082)(1-.085)(1-.088) \\
&= .288
\end{aligned}
$$

C

30 This is a typical exam question on commutation functions and life contingencies. You need to write a formula for N_{50}, and simplify it to calculate a value based on the l_x definition given.

$$N_{50} = \sum_{t=0}^{\omega-50} D_{50+t} = D_{50} + D_{51} + D_{52} + \ldots + D_{99}$$
$$= v^{50} l_{50} + v^{51} l_{51} + \ldots v^{99} l_{99}$$
$$= v^{50}[\, l_{50} + v l_{51} + \ldots + v^{49} l_{99}]$$

$$N_{50} = v^{50}[50 + 49v + 48v^2 + \ldots + 1v^{49}]$$

The usual trick is to multiply the equation by v^1, and subtract it from the prior equation.

$$v^1 N_{50} = v^{50}[\quad 50v + 49v^2 + \ldots + 2v^{49} + 1v^{50}]$$

$$(1-v) N_{50} = v^{50}[\, 50 - (v^1 + v^2 + \ldots + v^{49} + v^{50})]$$

$$N_{50} = \left(\frac{v^{50}}{1-v}\right)[50 - a_{\overline{50}|.04}]$$
$$= \frac{.1407}{.0385}[50 - 21.4822]$$
$$= 104.33$$

C

31 This is one of the relatively rare exam questions on the force of mortality. To work this problem, you must know the formula for the force of mortality at a fractional age. Based on the assumption of uniform distribution of decrements, you have

$$l_{x+t} = l_x - t(d_x) \qquad 0 \le t \le 1$$

$$\mu_{x+t} = \frac{q_x}{1-t(q_x)}$$

$$l_{65.5} = 975 = l_{65} - .5(d_{65}) \Rightarrow d_{65} = 50$$

$$q_{65} = 50/1000 = .05$$

$$\mu_{65.4} = \frac{q_{65}}{1-.4(q_{65})} = \frac{.05}{1-.4(.05)} = .05102$$

C

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