SoftwarePolish

Rick Groszkiewicz
2964 Nestle Creek Drive
Marietta, GA 30062-4857

Voice/fax (770) 971-8913

email: rickg@softwarepolish.com

SPRING 2014 EA-1
EXAM SOLUTIONS

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Revision History:

11/29/17	Revised solutions for problems 3 and 11
02/10/15	Revised solution for problem 16

2014 EA-1 Solutions

1 This is a fairly typical question on annuity calculations with unknown interest and for an unknown time period. The first step is writing down the payments on a time line diagram

There are many ways to write this series of payments. You can use differences between annuities, or sums of annuities. Based on the annuity factors given, it looks best to write a formula with $a_{\overline{n}|}$ and $a_{\overline{m}|}$

$$X = 1500\, a_{\overline{n}|} + 2500\, v^n a_{\overline{m}|} + 3500\, v^{n+m}\left(a_{\overline{n}|} + v^n a_{\overline{n}|}\right)$$

The only item you need is v^n. It is a bit tricky, but you can calculate the value without solving for n or m. The first step is solving for the interest rate:

$$a_{\overline{m}|} = 9.818 = \frac{1-v^m}{i} \quad \Rightarrow \quad i = (1-v^m)/9.818$$

$$= .785/9.818$$

and $v^m = .215$

$$= 8.00\%$$

(1) continued

Now you can use the formulas to solve for the value of v^n

$$a_{\overline{n}|} = \frac{1-v^n}{i}$$

$$8.559 = \frac{1-v^n}{.08} \Rightarrow v^n = 1-.08(8.559)$$
$$= .3153$$

$$X = 1500\, a_{\overline{n}|} + 2500\, v^n a_{\overline{n}|} + 3500\, v^{n+m}(a_{\overline{n}|} + v^n a_{\overline{n}|})$$
$$= 1500(8.559) + 2500(.3153)(8.559)$$
$$+ 3500(.3153)(.215)(8.559 + .3153(8.559))$$
$$= 12{,}839 + 7{,}739 + 2{,}671$$
$$= 23{,}248$$

E

2 This is a typical exam question on the idea of mortality improvement. This time the question tells you how to determine the projected q_x after t years of mortality improvement: $q_x(.95)^t$.

The l_x table follows De Moivre's law:

$l_x = 1000(110-x)$ <u>prior to mortality projection</u>

$l_{50} = 1000(60)$

$l_{51} = 1000(59)$ $\quad d_{50} = 1000$ $\quad q_{50} = 1/60$

$l_{52} = 1000(58)$ $\quad d_{51} = 1000$ $\quad q_{51} = 1/59$

$l_{53} = 1000(57)$ $\quad d_{52} = 1000$ $\quad q_{52} = 1/58$

You can write a formula for the net single premium, ignoring the mortality projection:

$$P = 10{,}000\left(v^1 q_{50} + v^2\,{}_2q_{50} + v^3\,{}_3q_{50}\right)$$
$$= 10{,}000\left(\frac{1-q_{50}}{1+i} + \frac{(1-q_{50})(1-q_{51})}{(1+i)^2} + \frac{(1-q_{50})(1-q_{51})(1-q_{52})}{(1+i)^3}\right)$$

Now you need to allow for the .95 adjustments for future mortality improvements after age 50:

$$X = 10{,}000\left(\frac{1-1/60}{1.05} + \frac{\left(\frac{59}{60}\right)\left(1-\frac{.95}{59}\right)}{(1.05)^2} + \frac{\frac{59}{60}\left(1-\frac{.95}{59}\right)\left(1-\frac{(.95)^2}{58}\right)}{(1.05)^3}\right)$$
$$= 10{,}000\left(.9365 + .9833(.9839)/1.1025 + .9833(.9839)(.9844)/1.1576\right)$$
$$= 26{,}368$$

B

3 This is a typical exam question on select and ultimate mortality. The key is using the relationship between a life insurance and an annuity due.

$p_x = .96 \quad q_x = .04$

$q_{[x]} = .75(q_x)$

$= .03$

$A_x = 1 - d\ddot{a}_x$

$A_{[x]} = 1 - d\ddot{a}_{[x]} \Rightarrow X = \ddot{a}_{[x]} = \dfrac{1 - A_{[x]}}{d}$

It is a bit confusing that X is the unknown, and x is the participant's current age. Now you must derive the value of $A_{[x]}$ based on the given value for A_x

$A_x = vq_x + vp_x A_{x+1}$

$A_{[x]} = vq_{[x]} + vp_{[x]} A_{x+1}$ with only 1 select year

$A_{x+1} = (A_x - vq_x)/vp_x$

$= [.60 - (.04/1.05)]/(.96/1.05)$

$= .6146$

$A_{[x]} = (.03/1.05) + (1-.03)(.6146)/1.05$

$= .5963$

$X = \ddot{a}_{[x]} = (1-.5963)/[(.05)(1.05)^{-1}] = 8.4771$ B

4 This is a typical exam question on annuity formulas, specifically the relation between an annuity due and an annuity immediate.

There are two methods to work this problem. The brute force approach is to solve for the interest rate, and calculate the value of X directly.

The other approach involves writing down some annuity formulas on a time line diagram, and deriving a formula for the numerator and denominator of the fraction that defines the value of X.

The diagram above shows $s_{\overline{n}|}$. Without the final payment, you could evaluate $s_{\overline{n-1}|}$ at time n-1. As an alternative, $s_{\overline{n-1}|}$ represents the accumulated value of the payments from time 2 through time n. The difference between $s_{\overline{n}|}$ and $s_{\overline{n-1}|}$ is the accumulated value of the payment at time 1, which is $(1+i)^{n-1}$.

You can easily derive the value of $\ddot{s}_{\overline{n}|} - \ddot{s}_{\overline{n-1}|}$ as well:

$$s_{\overline{n}|} - s_{\overline{n-1}|} = (1+i)^{n-1} \Rightarrow \ddot{s}_{\overline{n}|} - \ddot{s}_{\overline{n-1}|} = (1+i)^{n-1}(1+i) = (1+i)^n$$

(4) Now use the prior formulas to determine the value of X:

$$X = \frac{(s_{\overline{19}|} - s_{\overline{18}|})^2}{(\ddot{s}_{\overline{9}|} - \ddot{s}_{\overline{8}|})^3} = \frac{(1+i)^{18 \cdot 2}}{(1+i)^{9 \cdot 3}} = \frac{(1+i)^{36}}{(1+i)^{27}} = (1+i)^9$$

You can use the time line diagram for the values of $\ddot{a}_{\overline{n}|}$ and $\ddot{a}_{\overline{n-1}|}$. But it is just as easy to write out the series of payments:

$$\ddot{a}_{\overline{n}|} = 1 + v^1 + \ldots + v^{n-2} + v^{n-1}$$
$$\ddot{a}_{\overline{n-1}|} = 1 + v^1 + \ldots + v^{n-1}$$

$$\ddot{a}_{\overline{n}|} - \ddot{a}_{\overline{n-1}|} = v^{n-1} \Rightarrow \ddot{a}_{\overline{10}|} - \ddot{a}_{\overline{9}|} = .50 = v^9$$
$$(1+i)^9 = 1/.50$$
$$= 2.00$$

$\therefore X = 2.00$ (D)

For the "brute force" solution, you can solve for the interest rate based on $v^9 = .50$. The resulting value of i is 8.006%. You can use your calculator to calculate the value of X directly:

$$X = \frac{(s_{\overline{19}|8.006} - s_{\overline{18}|8.006})^2}{(\ddot{s}_{\overline{9}|8.006} - \ddot{s}_{\overline{8}|8.006})^3}$$
$$= \frac{(41.4720 - 37.4720)^2}{(13.4907 - 11.4907)^3} = \frac{16}{8} = 2.0$$

5 This is a typical exam question on mortality laws, as well as probability calculations. You are given $S(x)$, which is the probability of survival to age x, for ages under 25. You are given the formula for l_x for ages 25 and over - which you can use to calculate the probability of survival.

To work this problem, you need to separate out calculations below and above age 25. The problem defines X as the symbol ${}_{30|10}q_{10}$. You must be able to interpret this correctly. It is the 10 year probability of death, after 30 years, for a life that is currently age 10:

$$X = {}_{30|10}q_{10}$$
$$= ({}_{30}p_{10})({}_{10}q_{40})$$

$${}_{30}p_{10} = {}_{15}p_{10}\,({}_{15}p_{25}) = \frac{l_{25}}{l_{10}}\left(\frac{l_{40}}{l_{25}}\right)$$

$${}_{10}q_{40} = 1 - {}_{10}p_{40} = 1 - \frac{l_{50}}{l_{40}}$$

use $S(x)$ below age 25

use l_x for 25+ older

$$\frac{l_{25}}{l_{10}} = \frac{S(25)\,l_0}{S(10)\,l_0} = \frac{1 - 25/250}{1 - 10/250}$$
$$= .9375$$

(5) $${}_{15}p_{25} = \frac{l_{40}}{l_{25}} = \frac{1000(100-40)}{1000(100-25)} = \frac{60}{75}$$

$${}_{10}p_{40} = \frac{l_{50}}{l_{40}} = \frac{1000(100-50)}{1000(100-40)} = \frac{50}{60}$$

$$\begin{aligned} X &= ({}_{30}p_{10})({}_{10}q_{40}) \\ &= ({}_{15}p_{10})({}_{15}p_{25})(1-{}_{10}p_{40}) \\ &= .9375(60/75)(1-50/60) \\ &= .1250 \end{aligned}$$

B

6 This is a typical exam question on increasing (or decreasing) annuities, as well as perpetuities. The first step is writing down the payments on a time line diagram. Let Y be the decreasing annuity immediate, and let Z be the increasing perpetuity

for Y

for Z

The present value of each series of payments is equal to X. You need to know the general formula for increasing annuity

$$X = P a_{\overline{n}|i} + Q\left(\frac{a_{\overline{n}|} - n v^n}{i}\right)$$

For annuity Y, we have $P = 10$, $Q = -1$ and $n = 10$.

$$X = 10 a_{\overline{10}|i} - \left(\frac{a_{\overline{10}|} - 10v^{10}}{i}\right) = 10\left(\frac{1-v^{10}}{i}\right) - \left(\frac{a_{\overline{10}|} - 10v^{10}}{i}\right)$$

$$= \frac{10 - a_{\overline{10}|}}{i}$$

The increasing perpetuity Z is actually the sum of a 10 year increasing annuity and a perpetuity.

(6) BUT - you don't need to write a formula for the present value of Z. Instead, look a what you get when you add the payments from both Y and Z together

The result is a perpetuity of 11, which has a present value of $\frac{11}{i}$. This is also equal to 2X, since each of Y and Z has a present value equal to X.

$$2X = \frac{11}{i} \Rightarrow X = \frac{5.5}{i} = \frac{10 - a_{\overline{10}|}}{i}$$

$$5.5 = 10 - a_{\overline{10}|}i$$

$$a_{\overline{10}|}i = 4.5$$

Now you can solve for the interest rate using your calculator, which gives $i = 17.9639\%$.

$$X = \frac{5.5}{.17963} = 30.62$$

D

7. This is a typical exam question on renegotiated loan provisions. The key idea is that at any time during the life of the loan, the present value of the remaining loan payments must be equal to the outstanding balance of the loan.

Let Y be the original loan payment

$$Y = \frac{200{,}000}{a_{\overline{360}|.5\%}} = 1{,}199.10$$

Note that you are given the interest as a nominal annual rate, which is convertible monthly.

After 60 payments, you can determine the outstanding loan balance

$$O/S\ balance = 1{,}199.10\,(a_{\overline{300}|.5\%})$$
$$= 186{,}109$$

Now an additional payment of 50,000 is made, and the payment is redetermined for 10 more years. The interest rate is now .05/12 per month

$$X = \frac{(186{,}109 - 50{,}000)}{a_{\overline{120}|.4167\%}}$$
$$= 1{,}443.64$$

B

8 This is one of the simpler multiple decrement questions on the exam recently. The key is being able to correctly interpret the second probability given:

$$1|q_x^{(1)} = p_x^{(T)} q_{x+1}^{(1)}$$

$$.2292 = \left[1 - q_x^{(1)} - q_x^{(2)}\right] q_{x+1}^{(1)}$$

The problem asks for the value of $q_x^{(1)}$, which you can do easily – since you are given values for the other two items in the prior formula.

$$1 - q_x^{(1)} - q_x^{(2)} = \frac{.2292}{q_{x+1}^{(1)}}$$

$$1 - q_x^{(2)} - \frac{.2292}{q_{x+1}^{(1)}} = q_x^{(1)}$$

$$1 - .1250 - \frac{.2292}{.3333} = q_x^{(1)} \Rightarrow q_x^{(1)} = .1873$$

B

9 This is a very short question on callable bonds, which is unusual. The first step is writing the standard formula for the price of a bond:

$$X = Fr(a_{\overline{n}|i}) + Cv^n$$

$$= .06(1000)\,a_{\overline{n}|.05} + 1000(1.05)^{-n}$$

The key concept is knowing that the coupon rate is "more valuable" than the yield rate – so the more coupons that are paid, the better the result for the buyer. The worst scenario for the buyer is the earliest possible redemption, since fewer coupons will be paid.

You should calculate the price based on the bond being called at the earliest date. This is the maximum amount the bond purchaser would pay, and still be guaranteed to earn at least 5%:

$$X = 60(a_{\overline{10}|.05}) + 1000(1.05)^{-10}$$
$$= 60(7.7217) + 613.91$$
$$= 1{,}077.22$$

A

If you want, you could also calculate the bond price based on redemption after 15 years (as a check). This gives a bond price of 1,103.80.

10 This is a typical exam question on the various "mortality laws", such as the assumption of constant force of mortality, or uniform distribution of decrements.

The key is carefully interpreting $q_{x+.5}$, and knowing definition for partial year decrement calculations. The question states that mortality is based on uniform distribution of decrements.

$$t q_x = t(q_x)$$

$$y q_{x+t} = \frac{y(q_x)}{1-t(q_x)}$$

$$X = q_{x+.5} \quad \text{(one year probability of death)}$$
$$= (.5 q_{x+.5}) + (1 - .5 q_{x+.5})(.5 q_{x+1})$$

You must separately calculate the probability of death from age x+.5 to x+1, and also allow for survival to age x+1 followed by death in the next .5 year.

$$.5 q_{x+.5} = .5(q_x)/(1-.5 q_x) = .5(.40)/(1-.5(.40))$$
$$= .20/.80 = .25$$

$$X = .25 + (1-.25)[(.5)(.500)] \qquad .5 q_{x+1}$$
$$= .25 + .75(.25) = .4375 \qquad \text{C}$$

11 This is a fairly typical exam question on life annuity calculations. You must really understand commutation functions to work this. It is not really difficult, but there is quite a lot of messy arithmetic at the end.

$$X = a_{64:65:\overline{2}|}$$

This is a two year temporary joint life annuity. You should write this in basic terms, using the probability of survival

$$X = v\,{}_1p_{64}({}_1p_{65}) + v^2\,{}_2p_{64}({}_2p_{65})$$

The problem gives you the N_x commutation functions, which you can use to derive the D_x commutation factors. And those will give you the value of vp_x:

$$D_x = v^x l_x \qquad N_x = \sum_{t=0}^{\omega} D_{x+t} \qquad N_x - N_{x+1} = D_x$$

$$\frac{D_{x+1}}{D_x} = v p_x$$

$N_{64} - N_{65} = D_{64} = 294{,}298 - 263{,}044 = 31{,}254$

$N_{65} - N_{66} = D_{65} = 263{,}044 - 234{,}434 = 28{,}610$

$N_{66} - N_{67} = D_{66} = 234{,}434 - 208{,}299 = 26{,}135$

$N_{67} - N_{68} = D_{67} = 208{,}299 - 184{,}481 = 23{,}818$

(11) $X = vp_{64}(p_{65}) + v^2(p_{64}p_{65})(p_{65}p_{66})$

$$\frac{D_{x+1}}{D_x} = vp_x$$

$$vp_{64} = \frac{D_{65}}{D_{64}} = .9154$$

$$vp_{65} = \frac{D_{66}}{D_{65}} = .9135 \Rightarrow p_{65} = 1.07\left(\frac{D_{66}}{D_{65}}\right)$$

$$vp_{66} = \frac{D_{67}}{D_{66}} = .9113 \qquad = .9774$$

$$X = vp_{64}(p_{65}) + vp_{64}(p_{65})(p_{65})(vp_{66})$$

$$= vp_{64}(p_{65})[1 + p_{65}(vp_{66})]$$

$$= .9154(.9774)[1 + .9774(.9113)]$$

$$= 1.6918$$

C

12. This is a typical exam question on calculating investment return. This is shorter than some questions, since you do not have to calculate the time-weighted return. The first step is writing down the market values and cash flows on a time-line diagram

cash flows		-10,000	50,000 -10,000	-10,000	-10,000
	1-1	4-1	7-1	10-1	12-31
MV	100,000				130,000

Now you can write a formula based on simple interest to accumulate everything to the end of the year

$$100{,}000\left(1+\frac{12}{12}i\right)-10{,}000\left(1+\frac{3}{4}i\right)+40{,}000\left(1+\frac{2}{4}i\right)-10{,}000\left(1+\frac{i}{4}\right)$$

$$-10{,}000=130{,}000$$

Now you can collect all the terms involving i, and rewrite the result

$$100{,}000-10{,}000+40{,}000-10{,}000$$
$$+i\left(100{,}000-10{,}000\left(\frac{3}{4}\right)+40{,}000\left(\frac{2}{4}\right)-10{,}000\left(\frac{1}{4}\right)\right)=140{,}000$$

$$i=\frac{20{,}000}{100{,}000\left(\frac{4}{4}\right)-10{,}000\left(\frac{3}{4}\right)+40{,}000\left(\frac{2}{4}\right)-10{,}000\left(\frac{1}{4}\right)}=18.18\%$$

B

This is an exposure formula. The numerator is total interest earned. The denominator is the weighted exposure.

13 This is a routine question on expectation of life. If you know the definition for e_x, there is nothing tricky about the solution. You must know the definition for ${}_2q_{63} = 1 - p_{63}(p_{64})$. Then you can derive the p_x values based on e_x and e_{x+1}.

$$
\begin{aligned}
e_x &= {}_1p_x + {}_2p_x + {}_3p_x + \dots \\
&= {}_1p_x\,(1 + p_{x+1} + {}_2p_{x+1} + \dots) \\
&= p_x\,(1 + e_{x+1})
\end{aligned}
$$

$$p_x = e_x / (e_{x+1} + 1.0)$$

$$p_{63} = \frac{e_{63}}{1 + e_{64}} \qquad p_{64} = \frac{e_{64}}{e_{65} + 1}$$

$$
\begin{aligned}
{}_2q_{63} &= 1 - p_{63}(p_{64}) \\
&= 1 - \left(\frac{e_{63}}{1 + e_{64}}\right)\left(\frac{e_{64}}{1 + e_{65}}\right) \\
&= 1 - \left(\frac{9.5}{1 + 9.0}\right)\left(\frac{9.0}{1 + 8.5}\right) \\
&= .100
\end{aligned}
$$

B

14 This bond question has a slight twist. You are given spot rates, which you should assume are the basis of the yield required by the purchaser of the bond.

You need to know how to use the spot rates. Each spot rate given for a time t is the yield to maturity for a zero coupon bond that matures at time t. You should use the t=3 spot rate to discount all cash flows from t=3 back to today.

The first step is writing down the usual bond price formula. Then you need to write it in more detail, since you can't value the annuity directly:

$$X = Fr(a_{\overline{n}|i}) + Cv^n$$
$$= Fr(v^1 + v^2 + \ldots + v^n) + Cv^n$$

Now you can write the price of the bond based on the spot rates and 6% coupon rate:

$$X = .06(1000)\left[(1.07)^{-1} + (1.08)^{-2} + (1.09)^{-3}\right] + 1000(1.09)^{-3}$$
$$= 60 * \left[.9346 + .8573 + .7722\right] + 1000(.7722)$$
$$= 926.03$$

C

15 This is a typical exam problem involving spot rates and the yield curve. The key is figuring out what they are asking for – the year 3 forward rate.

The first step is writing down the cash flows on a time line diagram:

Present value is $950 = 500(\ ?\ ) + 700(1.07)^{-4}$

The problem gives you the 2 year spot rate as 5.0%, and the 4 year spot rate as 7.0%. What you need to discount the \$500 at time 3 is the 3 year spot rate.

The definition of the "forward rate" is that it represents the rate of return for a single year, under the yield curve. If j is the 3 year spot rate, you have

$$(1+j)^{-3} = (1.05)^{-2}(1+x)^{-1}$$ use this to discount \$500 at $t=3$

$$950 = 500(1.05)^{-2}(1+x)^{-1} + 700(1.07)^{-4}$$

$$500(1.05)^{-2}(1+x)^{-1} = 950 - 700(1.07)^{-4}$$

$$\frac{500(1.05)^{-2}}{950-700(1.07)^{-4}} = 1+x = 1.09025$$

$$x = 9.025\%$$

D

16 This is a multiple decrement problem that is slightly unusual. The catch is the subtle difference in the withdrawal decrements versus the mortality decrement. It seems that the withdrawal decrement occurs instantaneously at mid-year, but the mortality is distributed throughout the year.

I'm not sure how to modify the standard multi-decrement formulas to allow for the "mid-year thing". So I'll work the problem assuming that it must be pretty close to using a U.D.D. assumption for withdrawal as well as mortality.

Instead of working this as a three decrement problem, set aside the retirement decrement for now. 20% of the population retires at the start of the year, and the remaining 80% are exposed to both mortality and termination. I'll use the typical 2 decrement formulas:

$$q'^{(1)}_x \doteq q^{(1)}_x / \left(1 - \tfrac{1}{2} q'^{(2)}_x\right) \qquad q'^{(2)}_x \doteq q^{(2)}_x / \left(1 - \tfrac{1}{2} q'^{(1)}_x\right)$$

$$q^{(1)}_x \doteq \frac{q'^{(1)}_x \left(1 - \tfrac{1}{2} q'^{(2)}_x\right)}{1 - \tfrac{1}{4} q'^{(1)}_x q'^{(2)}_x}$$

(16) $$q_{63}^{(d)} \doteq \frac{q_{63}^{\prime(d)}\left(1-.5\,q_{63}^{\prime(w)}\right)}{1-.25\,q_{63}^{\prime(d)}\,q_{63}^{\prime(w)}}$$

$$= \frac{.02\,(1-.5(.125))}{1-.25(.02)(.125)}$$

$$= .01876$$

(E) -005

Don't forget that only 80% of the population remains after the retirement decrement hits at the start of the year:

$$q_{63}^{(d)} = .80\,(.01876)$$
$$= .01501$$

(B)

There is another formula you can use, which is not approximate. It gives precise results both for the U.D.D. assumption, and for the constant force assumption:

$${}_tp_x^{\prime(i)} = {}_tp_x^{(\tau)\left[\frac{q_x^{(i)}}{q_x^{(\tau)}}\right]}$$ from the Bowers book

$$p_{63}^{(\tau)} = (1-.02)(1-.125)$$ ignoring retirement
$$= .8575$$

$$p_{63}^{\prime(d)} = 1 - q_{63}^{\prime(d)} = .8575^{\left[q_{63}^{(d)}/q_{63}^{(\tau)}\right]}$$

Time to do some log-based calculations!

(16) $(1-.02) = (.8575)^{(q_{63}^{(d)}/q_{63}^{(T)})}$

$$.980^{(1-.8575)} = (.8575)^{q_{63}^{(d)}}$$

$$.1425(\log .98) = q_{63}^{(d)}(\log .8575)$$

$$q_{63}^{(d)} = .1425\left(\frac{\log .980}{\log .8575}\right)$$

$$= .01873 \qquad \text{(not quite)}$$

Again, don't forget to reflect that 20% of the population is assumed to retire at the begining of the year:

$$q_{63}^{(d)} = .80(.01873)$$
$$= .01498 \qquad \text{(B)}$$

Both of these solutions should really be rounded off to .0150. The reason is that the data given in the problem only has three significant digits. The answer ranges are slightly defective as a result - they normally are expressed using the same number of significant digits as the data in the problem.

ALTERNATE-BAD solution on following pages

(16) I originally tried to work this problem in a way that accurately reflects the idea of "instantaneous" mid year withdrawals. It almost works - close, but no cigar!

The approach is calculating the mortality based on exposures. The people exposed to withdrawal should only reflect the first six months of age. To calculate $q_{63}^{(d)}$, you have to determine the deaths in the first six months of age separately from the deaths in the last six months of age.

I'll start by assuming $l_{63} = 1000$ (any number can be used, since it affects both denominator and numerator for the q_x fraction). Retirements occur at the beginning of the year, so there are 800 participants exposed to mortality. Based on $q_{63}^{\prime(d)} = .02$, there should be 8 deaths in the first six months of age 63:

$$\left(\frac{6}{12}\right)(.02)(800) = 8$$

At mid-year there are 792 participants exposed to termination. At that point, there are 99 exits on .125(792). For the last six months of age, there are 693 participants exposed to the force of mortality:

792 - 99 withdrawals = 693

(16) Based on $q_{63}^{(d)} = .02$, there are less than 7 deaths in the last six months of age 63:

$$\left(\frac{6}{12}\right)(.02)(693) = 6.93 \text{ deaths}$$

Now you can calculate $q_{63}^{(d)}$ as a simple ratio

$$q_{63}^{(d)} = \frac{8.0 + 6.93}{1000}$$

$$= .01493$$

(A) - 0085

Unfortunately, this result falls in the wrong answer range - just barely!!

The only "magic adjustment" that could increase the value slightly would be this formula:

$$_{1-t}q_{x+t} = \frac{(1-t)\,q_x}{1-t(q_x)} \quad \text{for U.D.D.}$$

If you use this to adjust the number of deaths in the last 6 months, it produces an answer of .01500. But I'm not sure this is correct, or that it makes any sense. The point of this problem is calculating $q_x^{(d)}$ when you are given $q_x^{(d)}$. It seems wrong to use the U.D.D. adjustment to calculate $q_x^{(d)}$.

(16) For the sake of completeness, here is what happens if I make the incorrect adjustment.

$$ {}_{.5}q_{63.5} = \frac{(1-.5)\,q_{63}}{1-({}_{.5}q_{63})} $$ this is correct

$$ {}_{.5}q'_{63.5} = \frac{(1-.5)\,q_{63}}{1-({}_{.5}q_{63})} $$ this is incorrect!

$$ = \frac{.5(.02)}{1-.5(.02)} $$

$$ = 1.01\% $$

Now the deaths in the second half of age 63 are slightly higher: $1.01\%(693) = 7.00$ deaths

$$ q_{63}^{(d)} = \frac{8.0+7.0}{1000} $$

$$ = .0150 $$

B

17. This is a typical exam question on life insurance and premium calculations. The first step is writing down a formula for the premium (I'll use P), and it is equal to the present value of the death benefits:

$$P = 10{,}000A_{40} + .5P\left[1.05v^1 q_{40} + (1.05)^2 v^2 {}_{1|}q_{40} + \ldots + (1.05)^{10} v^{10} {}_{9|}q_{40}\right]$$

$$= 10{,}000A_{40} + .5P\left[q_{40} + {}_{1|}q_{40} + {}_{2|}q_{40} + \ldots + {}_{9|}q_{40}\right]$$
$$= 10{,}000A_{40} + .5P\left[1 - {}_{10}p_{40}\right]$$

It should make sense that the sum of probabilities of death for each of the next 10 years is simply ${}_{10}q_{40}$. You need to use the factors given in the problem to derive the value of ${}_{10}q_{40}$. The ${}_nE_x$ factor is the pure endowment, which pays \$1 at age $x+n$ if the person is alive at that age.

$${}_5E_{40} = v^5 {}_5p_{40}$$
$${}_5E_{45} = v^5 {}_5p_{45}$$
$${}_5E_{40}({}_5E_{45}) = v^{10} {}_{10}p_{40} \Rightarrow {}_{10}p_{40} = (1+i)^{10}({}_5E_{40})({}_5E_{45})$$
$$= (1.05)^{10}(.71823)(.71230)$$
$$= .83333$$

$$P = 10{,}000A_{40} + .5P(.16667)$$
$$P - .5P(.16667) = 10{,}000(.31549)$$
$$P = \frac{10{,}000(.31549)}{1 - .5(.16667)} = 3{,}441.71$$

B

18 If this looks familiar, you have probably worked it already. This question is <u>identical</u> to problem 18 on the 2007 EA-1 exam!

The key to working the problem is knowing how to write out the series of payments for a level income benefit:

The initial benefit is X, starting at age 55. At age 62, the Social Security benefit of \$1400 is paid. The plan benefit reduces by 1400 at 62, so the total amount received by the participant is level.

PV of normal form = PV of optional form

$$12(750)\ddot{a}_{55}^{(12)} = 12(X)\left(\ddot{a}_{55:\overline{7}|}^{(12)}\right) + 12(X-1400)\,{}_{7|}\ddot{a}_{55}^{(12)}$$

$$750\,\ddot{a}_{55}^{(12)} = X\left(\ddot{a}_{55}^{(12)} - {}_{7|}\ddot{a}_{55}^{(12)}\right) + (X-1400)\,{}_{7|}\ddot{a}_{55}^{(12)}$$

$$= X\ddot{a}_{55}^{(12)} - 1400\frac{D_{62}}{D_{55}}\ddot{a}_{62}^{(12)}$$

$$750 = X - 1400\left(\frac{D_{62}\ddot{a}_{62}^{(12)}}{D_{55}\ddot{a}_{55}^{(12)}}\right)$$

$$X = 750 + 1400\left({}_7E_{55}\right)\left(\ddot{a}_{62}^{(12)}/\ddot{a}_{55}^{(12)}\right)$$

$$= 750 + 1400(.656)(12.218/13.728) = 1567.38$$

C

19 This is a typical exam question on calculation of duration. Duration is the weighted average time of payment for a series of cash flows. The weight is actually the present value of each payment:

$$\bar{d} = \frac{\sum t(v^t R_t)}{\sum v^t R_t}$$

The formula above is the regular duration. The problem asks for the modified duration $X = \frac{\bar{d}}{1+i}$. One minor complication is that the interest rate and payments are on a monthly basis. But duration is actually defined on an annual basis – I'll divide by 12 at the end of the calculations.

The mortgage has 20 years of monthly payments, for 240 in total. You do not need to calculate the amount of the monthly payment, since it is in both the numerator and denominator of the fraction.

The interest rate is given as 4.5%, convertible on a monthly basis. The monthly rate is 4.5%/12 = .375%.

$$\bar{d} = \frac{P\,mat\,[1\cdot v^1 + 2\cdot v^2 + 3\cdot v^3 + \ldots + 240 v^{240}]}{P\,mat\,[v^1 + v^2 + v^3 + \ldots + v^{240}]}$$

$$= I a_{\overline{240}|.375\%} / a_{\overline{240}|.375\%}$$

(19) The key to working most duration problems is knowing the formula for an increasing annuity.

$$I a_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n v^n}{i}$$

$$I a_{\overline{240}|.375} = \frac{(1.00375)\ddot{a}_{\overline{240}|.375} - 240(1.00375)^{-240}}{.00375}$$

I use the HP-12C calculator, which has a setting to calculate immediate annuities versus annuities due. Most calculations for EA-1 require immediate annuities, so I leave the calculator set to calculate them. When I need an annuity due, I simply use the standard identity $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$.

$$I a_{\overline{240}|.375} = \frac{1.00375(158.0654) - 240(.40725)}{.00375}$$

$$= 16{,}244.55$$

$$\bar{d} = 16{,}244.55 / 158.0654$$

$$= 102.7711$$

$$\frac{\bar{d}}{1+i} = 102.3871 \quad \text{on a monthly basis}$$

$$\text{Annual duration} = 102.3871/12$$

$$= 8.532$$

C

20 This is a typical exam question on accumulated fund values. Smith deposits 5% of pay at the end of each year, and their salary increases at 3% annually.

The first step is writing down the fund deposits on a time line diagram. Then you can construct the formula for the accumulated value at 01/01/2035:

$$X = 2500\left[(1.06)^{20} + (1.03)^1(1.06)^{19} + \ldots + (1.03)^{19}(1.06)^1 + (1.03)^{20}\right]$$

Note that the sum of the exponents in each term is equal to 20. To simplify this type of summation, the standard "trick" is to factor out either the first term or the last term of the series.

$$X = 2500(1.06)^{20}\left[1 + \frac{1.03}{1.06} + \ldots + \left(\frac{1.03}{1.06}\right)^{20}\right]$$

$$= 2500(1.06)^{20}\,\ddot{a}_{\overline{21}|j} \quad \text{where } 1+j = 1.06/1.03 \Rightarrow j = 2.913\%$$

$$= 2500(3.2071)(1.02913)\,a_{\overline{21}|2.913\%}$$

$$= 128{,}272$$

C

21 This is a potentially tricky question on joint and last survivor annuities. If you are comfortable using the idea of a reversionary annuity, it makes the solution much easier. A reversionary annuity is payable to one person (age x) after the death of another person (age y):

$a_x - a_{xy}$ pays \$1 to x after the death of y

The problem gives three different amounts. The easiest one to value is \$1000 while all three are alive: $1000\, a_{64:64:64}$. The next easiest is \$750 payable while exactly two annuitants are alive:

$$PV = 750\left(a_{64:64} - a_{64:64:64}\right)(3)$$

The difference between the two annuity values is the reversionary annuity. It pays 750 to two annuitants after the death of the third annuitant. The factor of three allows for the fact that there are three cases, based on which of the participants is dead.

The final present value pays 500 while only one participant is alive. This means that both of the other two participants are dead:

$$PV = 500\left(a_{64} - a_{\overline{64:64:64}}\right)(3)$$

(21) $PV = 500(a_{64} - a_{64:\overline{64:64}})(3)$

The second annuity is a joint and last survivor annuity between the one participant being paid (who is alive) and the other two participants – both of whom must be dead. Since there are three participants who could be alive, you have the same factor of 3 to account for all the cases.

Now you can add together the three pieces and calculate the value of X:

$$X = 500(a_{64} - a_{64:\overline{64:64}})\,3$$
$$+ 750(a_{64:64} - a_{64:64:64})\,3$$
$$+ 1000\ a_{64:64:64}$$

I'll expand the joint and last survivor annuity, and line up all the annuity factors underneath each other

$$X = 1500(a_{64} - a_{64:64} - a_{64:64} + a_{64:64:64})$$
$$+2250(\quad a_{64:64} \quad - a_{64:64:64})$$
$$+1000(\quad a_{64:64:64})$$
$$= 1500\,a_{64} + (2250-3000)a_{64:64} + (2500-2250)a_{64:64:64}$$
$$= 1500(25) - 750(15) + 250(10)$$
$$= 28{,}750$$

C

22 This is a simplified claim question on the "mortality laws" such as constant force of mortality, uniform distribution of deaths, or the Balducci assumption. You only need to know how to determine one term under UDD: ${}_tq_x = t(q_x)$.

$$
\begin{aligned}
\overline{X} &= {}_{1.5}q_{60} \\
&= 1 - {}_{1.5}p_{60} \\
&= 1 - p_{60}({}_{.5}p_{61}) \\
&= 1 - (1-q_{60})(1-{}_{.5}q_{61}) \\
&= 1 - (1-.0200)(1-.5(.0220)) \\
&= 1 - (.9800)(.9890) \\
&= .03078
\end{aligned}
$$

B

One minor point of this problem is the extra zeroes after the mortality rates. They are necessary, since they allow you to assume there are three significant digits in the result calculated above.

If instead the data was given as $q_{60} = .02$ and $q_{61} = .022$, then you would only have two significant digits in the answer. The final result would not be in the same answer range!

23. This is a typical loan question on the loan amortization schedule. If you have memorized the typical loan amortization schedule, you can quickly write down the answer. I'll show how to construct the amortization schedule first:

Original loan balance $100\, a_{\overline{n}|i} = 100\dfrac{(1-v^{20})}{i}$

Pmt Num	Pmt amount	interest	principal	O/S loan
1	100	$100(1-v^{20})$	$100v^{20}$	$100(1-v^{19})/i$
2	100	$100(1-v^{19})$	$100v^{19}$	$100(1-v^{18})/i$
3	100	$100(1-v^{18})$	$\vdots$	
$\vdots$	$\vdots$	$\vdots$		

You are told that the loan expense is 3% of the interest portion of each payment: $.03(100)(1-v^{n}) = 3(1-v^{n})$.

$$\begin{aligned} X &= \text{sum of loan expenses} \\ &= 3(1-v^{20}) + 3(1-v^{19}) + \dots + 3(1-v^{1}) \\ &= 3\left[20 - a_{\overline{20}|.05}\right] \\ &= 22.61 \end{aligned}$$

A

24 This is a somewhat unusual present value of death benefit calculation. Smith purchases an annuity immediate of 10,000 per annum, plus a death benefit that continues for 10 years after Smith's death.

$$X = 10{,}000(\ddot{a}_{60} - 1.0) + 10{,}000(\ddot{a}_{\overline{10}|.05})A_{60}$$

You need to be careful that you define the death benefit as an annuity due. The life insurance A_{60} pays at the end of the year of death.

If Smith is alive, they receive a payment at the end of each year. In the year that Smith dies, the death benefit pays for 10 years, starting at the end of the year of death.

$$\begin{aligned} X &= 10{,}000\left(11.70 + \ddot{a}_{\overline{10}|.05}(1 - d\,\ddot{a}_{60})\right) \\ &= 10{,}000\left(11.70 + 1.05(7.7217)\left(1 - \frac{.05}{1.05}(12.70)\right)\right) \\ &= 149{,}045 \end{aligned}$$

C

25 If you feel that you have worked this problem before, you are correct! This problem is identical to 2009 exam question 24.

This is a typical exam question on actuarial equivalence and joint and survivor annuities. Since the annuities are actuarially equivalent, they all have the same present value.

In order to write present value formulas, I will assign age y to Smith and age z to Smith's surviving spouse.

PV #1 : $12(100)\ddot{a}_y^{(12)}$

PV #2: $12(94)\ddot{a}_y^{(12)} + 12(47)(\ddot{a}_z^{(12)} - \ddot{a}_{yz}^{(12)})$

PV #3: $12(X)\ddot{a}_y^{(12)} + 12(.75X)(\ddot{a}_z^{(12)} - \ddot{a}_{yz}^{(12)})$

Now you set the present values equal to each other, and discard the monthly factor of 12:

$$100\ddot{a}_y^{(12)} = 94\ddot{a}_y^{(12)} + 47(\ddot{a}_z^{(12)} - \ddot{a}_{yz}^{(12)})$$

$$100\ddot{a}_y^{(12)} = X\ddot{a}_y^{(12)} + .75X(\ddot{a}_z^{(12)} - \ddot{a}_{yz}^{(12)})$$

(25) It looks like you have too many unknowns to solve for X. But there is a standard technique in all the problems with J+S annuities – divide both sides by $\ddot{a}_{y}^{(12)}$, and use R to represent the ratio of $(\ddot{a}_{yz}^{(12)} - \ddot{a}_{y}^{(12)})/\ddot{a}_{y}^{(12)}$:

$$100 = 94 + 47R \Rightarrow R = .12766$$

$$100 = X + .75RX$$

$$= X + .75X(.12766)$$

$$X = \frac{100}{1 + .75(.12766)}$$

$$= 91.26$$

B

26 This is a simple insurance problem with a minor twist. Instead of giving you an interest rate to value the death benefit, this problem gives you the prices for several zero coupon bonds. This conforms to the idea of using the yield curve to value the death benefits.

The normal bond price formula includes coupon payments and the redemption of the bond:

$$P = Fr\,a_{\overline{n}|i} + Cv^n$$

If the bond has no coupons, then the price is equal to the discounted value of the redemption value.

$P = Cv^n$ for zero coupon bond

You should simply divide each of the bond prices given by 100, which gives the values of v^1, v^2 and v^3. Then use these values to calculate X:

$$\begin{aligned} X &= 1000\left(v^1 q_x + v^2\,{}_{1|}q_x + v^3\,{}_{2|}q_x\right) \\ &= 1000\left[.97(.20) + .93(1-.20)(.25) + .89(1-.20)(1-.25)(.30)\right] \\ &= 1000\left[.1940 + .1860 + .1602\right] \\ &= 540.20 \end{aligned}$$

B

27. This is a fairly short problem on select and ultimate mortality rates. The key is knowing the relationship between annuity values at successive ages. In general, you have this formula

$$\ddot{a}_x = v p_x \ddot{a}_{x+1} + 1.0$$

With a 1 year select period, the annuity at age x+1 is based on the ultimate mortality:

$$\ddot{a}_{[x]} = v p_{[x]} \ddot{a}_{x+1} + 1.0$$

$$\ddot{a}_{[70]} = v p_{[70]} \ddot{a}_{71} + 1.0 = v(1.013) p_{70} \ddot{a}_{71} + 1.0$$

$$\ddot{a}_{70} = v p_{70} \ddot{a}_{71} + 1.0 \quad \text{(not using select mortality)}$$

Based on the information given, you can't solve for the interest rate. But you can get the value of $v p_{70} \ddot{a}_{71}$, which is sufficient to work the problem.

$$9.8269 = v p_{70} \ddot{a}_{71} + 1.0 \Rightarrow v p_{70} \ddot{a}_{71} = 8.8269$$

$$\begin{aligned} \ddot{a}_{[70]} &= 1.013 (v p_{70} \ddot{a}_{71}) + 1.0 \\ &= 1.013 (8.8269) + 1.0 \\ &= 9.9416 \end{aligned}$$

C

28 This is a typical exam question on probability of survival. The key in most of these questions is writing everything in terms of l_x values. Then you may have to "study the alternatives" to find the answer - sometimes it can be hard to see.

40% probability that Smith and Brown survive 15 years:

$$.40 = {}_{15}p_{55}({}_{15}p_{40}) = \frac{l_{70}}{l_{55}}\left(\frac{l_{55}}{l_{40}}\right) = \frac{l_{70}}{l_{40}} = .40$$

44% probability that Jones dies before age 70:

$$.44 = 1 - {}_{25}p_{45} = 1 - \frac{l_{70}}{l_{45}} \Rightarrow \frac{l_{70}}{l_{45}} = .56$$

X = probability Brown dies before age 45

$$= {}_5q_{40}$$

$$= 1 - \frac{l_{45}}{l_{40}}$$

$$= 1 - \frac{l_{45}}{l_{40}}\left(\frac{l_{70}}{l_{70}}\right)$$

$$= 1 - \frac{l_{70}/l_{40}}{l_{70}/l_{45}}$$

$$= 1 - (.40/.56)$$

$$= .2857$$

B

This was not one of the trickier questions!

29 This is <u>not</u> a typical bond question. Instead of the usual calculation of the price of the bond, this question is more about interpreting the data/question.

You have one year bonds with 1000 redemption (and 1000 face value) and 8% coupon rate. If the bond defaults, the purchaser receives \$600. If the bond does not default, the purchaser receives \$1000 + .08(1000) = \$1080.

Based on the probability of default given, you must calculate how many bonds to purchase so the expected proceeds equal 2,000,000:

$$X\,(94\%(1080) + 6\%(600)) = 2{,}000{,}000$$

$$X = \frac{2{,}000{,}000}{1015.20 + 36.00}$$

$$= 1902.6$$

B

30 This is a loan problem with a minor twist. Instead of having the typical interest rate definition, you should assume the loan payments are based on the yield curve.

You are given three different spot rates. A spot rate is basically the yield for a zero coupon bond of the same maturity. You should discount all cash flows in the first 5 years at 4.0%, then use 6.0% for all cash flows in the next 5 years, and 6.5% thereafter.

$$100{,}000 = X\left[\ddot{a}_{\overline{5}|.04} + (\ddot{a}_{\overline{10}|.06} - \ddot{a}_{\overline{5}|.06}) + (\ddot{a}_{\overline{20}|.065} - \ddot{a}_{\overline{10}|.065})\right]$$

$$X = \frac{100{,}000}{1.05(a_{\overline{5}|4\%}) + 1.06(a_{\overline{10}|6\%} - a_{\overline{5}|6\%}) + 1.065(a_{\overline{20}|6.5\%} - a_{\overline{10}|6.5\%})}$$

$$= 100{,}000/(4.6299 + 3.3366 + 4.0786)$$

$$= 8{,}302$$

C

31 This is a typical exam question on the idea of select and ultimate mortality. The most important step is the first one - writing down the correct symbol for X. After you do that, you should write it in simplest terms, with $l_{[x]+t}$ and l_{x+t}. The last step is using the formulas given for both $l_{[x]+t}$ and l_{x+t}.

$$X = {}_{2|4}q_{[50]+2}$$

$$= \left({}_{2}p_{[50]+2}\right)\left(1 - {}_{4}q_{[50]+4}\right)$$

This expression does match the description for X!

$$X = \left(\frac{l_{[50]+4}}{l_{[50]+2}}\right)\left(1 - \frac{l_{58}}{l_{[50]+4}}\right)$$

Now you can use the formulas for $l_{[x]+t}$ and l_{x+t}:

$$X = \left(\frac{100-50-2}{100-50-1}\right)\left(1 - \frac{103-58}{100-50-2}\right)$$

$$= \frac{48}{49}\left(1 - \frac{45}{48}\right)$$

$$= .0612$$

C

32 This is a typical exam question on probability of survival calculations. You must write down formulas for all the items in terms of l_x values. Then you may have to think a bit to see how to get the value of X.

$${}_{15}p_{25} = .8108 = l_{40}/l_{25}$$

$$\begin{aligned} {}_{15|15}q_{25} = .2027 &= {}_{15}p_{25}\,({}_{15}q_{40}) \\ &= .8108\,(1 - {}_{15}p_{40}) \\ &= .8108\,(1 - l_{55}/l_{40}) \Rightarrow 1 - \frac{l_{55}}{l_{40}} = \frac{.2027}{.8108} \end{aligned}$$

$$\begin{aligned} {}_{10}q_{55} = .2222 &= 1 - {}_{10}p_{55} \\ &= 1 - l_{65}/l_{55} \end{aligned} \qquad \therefore \frac{l_{55}}{l_{40}} = .750$$

$$\begin{aligned} X = {}_{30|10}q_{25} &= {}_{30}p_{25}\,({}_{10}q_{55}) \\ &= \frac{l_{55}}{l_{25}}(.2222) \\ &= \left(\frac{l_{55}/l_{40}}{l_{25}/l_{40}}\right).2222 \\ &= .750\,(.8108)(.2222) \\ &= .1351 \end{aligned}$$

D