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SPRING 1996 EA-1A EXAM SOLUTIONS

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Revision History:

04/23/98	Enhanced problem 07	added detailed derivation
02/23/99	Enhanced problem 12	added faster method of solution
	Corrected problem 13	misstatement about handling 12/31/15 maturity date
02/19/00	Corrected problem 13	misstatement about Alternate price formula
	Enhanced problem 22	added faster method of solution
	Enhanced problem 23	added faster method of solution

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1. Problems on the force of mortality are rare on this exam. This is the first time the problem has been asked for a multiple decrement table.

The typical definition for a single decrement table is $\mu_x = -\frac{dl_x}{l_x}$. This problem asks for $\mu_{50}^{(2)}$, which is defined as $\mu_{50}^{(2)} = \frac{-d\, l_{50}^{(2)}}{l_{50}^{(T)}}$ based on Jordan's formula 14.22.

You are given formulas for $l_x^{(1)}$ and $l_x^{(2)}$. The key to the problem is how you should determine $l_x^{(T)}$. It should make sense that if $q_x^{(T)} = q_x^{(1)} + q_x^{(2)}$ and $l_x^{(1)} = \sum_{t=x}^{\omega} d_t^{(1)}$ $\quad l_x^{(2)} = \sum_{t=x}^{\omega} d_t^{(2)}$

then $l_x^{(T)} = l_x^{(1)} + l_x^{(2)}$

Age x	$l_x^{(1)}$	$l_x^{(2)}$	$l_x^{(T)}$
49	102	355	457
50	100	350	450
51	98	345	443

$$\mu_{50}^{(2)} = -\frac{1}{450}(-5) = \frac{1}{90}$$

$$100\,\mu_{50}^{(2)} = \frac{100}{90} = 1.11 \qquad \text{A}$$

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2. The first step is to draw a time diagram for the loan

Each repayment is reduced by 6% of the original loan amount of X. Since the loan interest is paid separately each year, this is equivalent to a new time diagram

In this diagram, you can accumulate the payments at 5% per annum to 1/1/99, and the value should equal X. This reflects the elimination of the separate interest payments towards the loan

$$X = 300 - .06X + 1.05(600 - .06X) + (1.05)^2(900 - .06X)$$
$$= (300 - .06X)s_{\overline{3}|.05} + 600(1.05)^2 + 300(1.05)$$
$$X + .06X s_{\overline{3}|.05} = 300 s_{\overline{3}|.05} + 600(1.05)^2 + 300(1.05)$$

$$X = \frac{300 s_{\overline{3}|.05} + 600(1.05)^2 + 300(1.05)}{1 + .06(s_{\overline{3}|.05})}$$
$$= \frac{300(3.1525) + 600(1.1025) + 300(1.05)}{1 + .06(3.1525)}$$
$$= (945.75 + 661.5 + 315)/1.1892$$
$$= 1616.42$$

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3. This is basically a life contingencies identity problem. The easy part of the problem is writing down the formula for the net single premium:

$$50{,}000\, A^{1}_{55:\overline{3}|} = 50{,}000\left(v\, {}_{0|}q_{55} + v^2\, {}_{1|}q_{55} + v^3\, {}_{2|}q_{55}\right)$$

You are told that ${}_{k|}q_{55} = (1.084)^k q_{55}$ for $k>0$,

but to evaluate this you must derive the value of q_{55}. The trick to the problem is to figure out how to do this using the fact that ${}_3p_{55} = .97057$:

$${}_3p_{55} = p_{55}\, p_{56}\, p_{57} = p_{55} p_{56}(1-q_{57}) = {}_2p_{55} - {}_{2|}q_{55}$$

Once you get here, you can apply similar logic to ${}_2p_{55}$:

$${}_2p_{55} = p_{55}p_{56} = p_{55}(1-q_{56}) = p_{55} - {}_{1|}q_{55} = 1 - {}_{0|}q_{55} - {}_{1|}q_{55}$$

$$\therefore\ {}_3p_{55} = 1 - {}_{0|}q_{55} - {}_{1|}q_{55} - {}_{2|}q_{55}$$

$$.97057 = 1 - (q_{55} - 1.084 q_{55} - (1.084)^2 q_{55})$$

$$q_{55} = .02943/(1 + 1.084 + (1.084)^2)$$

$$= .02943/3.2591 = .0090$$

$$50{,}000\, \ddot{a}^{1}_{55:\overline{3}|} = 50{,}000\left(\frac{q_{55}}{1.07} + \frac{{}_{1|}q_{55}}{(1.07)^2} + \frac{{}_{2|}q_{55}}{(1.07)^3}\right)$$

$$= 50{,}000(.0090)\left(\frac{1}{1.07} + \frac{1.084}{(1.07)^2} + \frac{(1.084)^2}{(1.07)^3}\right)$$

$$= 50{,}000\,\frac{(.0090)}{1.07}\left(1 + \left(\frac{1.084}{1.07}\right) + \left(\frac{1.084}{1.07}\right)^2\right)$$

$$= 50{,}000\left(\frac{.0090}{1.07}\right) S_{\overline{3}|}1.31\% = 1282.54$$

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4. Gross premium problems can be quite tricky. The key is to make sure you reflect all benefits and expenses, then combine terms carefully to simplify the final calculations:

PV of gross premiums	$G\ddot{a}_{45}$
PV of death benefit	$50{,}000 A_{45}$
75% first year commission	$.75G$
10% for years 2 through 5	$.10G a_{45:\overline{4}\rceil}$
5% for all later years	$.05G\ {}_{5|}a_{45}$
\$10 per 1000 first year expense	$10(50)$
\$2 per 1000 after first year	$2(50)a_{45}$
\$5 per 1000 for settlement cost	$5(50)A_{45}$

Next, rewrite all this, allowing for expenses over the life of the policy, with additional layers for years 2-5 and the first year:

$$G\ddot{a}_{45} = 50{,}000 A_{45} + .05G\ddot{a}_{45} + .05G\ddot{a}_{45:\overline{5}\rceil} + .65G + 100\ddot{a}_{45} + 400 + 250A_{45}$$

$$G = \frac{50{,}250 A_{45} + 100\ddot{a}_{45} + 400}{.95\ddot{a}_{45} - .05\ddot{a}_{45:\overline{5}\rceil} - .65}$$

$$= \frac{50{,}250(50/502) + 100(9500/502) + 400}{.95(9500/502) - .05\left(\frac{9500-7070}{502}\right) - .65}$$

$$= \frac{5005 + 1892 + 400}{17.98 - .2420 - .65}$$

$$= \frac{7297}{17.09}$$

$$= 427.10$$

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5 The time weighted return requires that you calculate the market value before and after each cash flow. The yield between cash flows is the ratio $\frac{\text{market value prior to cash flow}}{\text{market value after preceding cash flow}}$

cash flows →	1,200,000	1,450,000 −50,000	900,000 +100,000	1,400,000 +200,000	1,800,000	← market value before cash flow
	1/1/96	4/1/96	8/1/96	11/1/96	1/1/97	
	1,200,000	1,400,000	1,000,000	1,600,000	1,800,000	← market value after cash flow

Time weighted $1+i = \frac{1,450}{1,200}\left(\frac{900}{1,400}\right)\left(\frac{1,400}{1,000}\right)\left(\frac{1,800}{1,600}\right) = 1.2234 \Rightarrow i = 22.34\%$

The dollar weighted yield can be calculated using either simple or compound interest. You should equate the accumulated value of all cash flows and the initial asset value to the ending market value:

$$1,800,000 = 1,200,000(1+i) - 50,000\left(1+\tfrac{9}{12}i\right) + 100,000\left(1+\tfrac{5}{12}i\right) + 200,000\left(1+\tfrac{2}{12}i\right)$$

The problem is set up to trick you into using fractions of 6/12 and 3/12; based on 7/1/96 and 10/1/96, which is wrong!

$$350,000 = i\left[\tfrac{12}{12}(1,200,000) - \tfrac{9}{12}(50,000) + \tfrac{5}{12}(100,000) + \tfrac{2}{12}(200,000)\right]$$

$$i = 350,000 / [1,200,000 - 37,500 + 41,667 + 33,333]$$

$$= 350,000 / 1,237,500$$

$$= 28.28\%$$

The absolute difference in the two rates is 28.28% − 22.34%, or 5.94%

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6. The first step is to write expressions for each of the three annuity contracts

#1: $19{,}000 = 2{,}000\ {}_{10|}\ddot{a}_{40:\overline{20|}} = 2{,}000\left(\frac{N_{50}-N_{70}}{D_{40}}\right)$

#2: $4{,}400 = 500\ \ddot{a}_{70} = 500\left(\frac{N_{70}}{D_{70}}\right)$

#3: $X = 1{,}000\ \ddot{a}_{50} = 1{,}000\left(\frac{N_{50}}{D_{50}}\right)$

Since you are given the values of D_x for each of the three denominators, you can solve for N_{70} using the formula for contract #2. Next, you can solve for N_{50} using the formula for contract #1. Finally, you can use that to derive the value of X

#2: $4{,}400 = 500\left(\frac{N_{70}}{D_{70}}\right)$ $\therefore N_{70} = 8.8(D_{70})$
$= 8.8(3{,}000)$
$= 26{,}400$

#1: $19{,}000 = 2{,}000\left(\frac{N_{50}-26{,}400}{D_{40}}\right)$

$\frac{19{,}000\,D_{40}}{2{,}000} + 26{,}400 = N_{50} = 9.5(6{,}000) + 26{,}400$
$= 83{,}400$

#3: $X = 1000\left(\frac{N_{50}}{D_{50}}\right) = \left(\frac{83{,}400}{5{,}000}\right)1{,}000$
$= 16{,}680$

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7. This is a "retirement income policy" where the death benefit equals the return of premiums with interest. If you write the summation formula for the reserve that includes the pre-retirement death benefits, you will discover that all the mortality terms cancel out. You get exactly the same results by ignoring the mortality decrements prior to age 65.

At age 65, the accumulated value of all premiums should equal the present value of future benefits:

$$12X\,\ddot{a}_{65}^{(12)} = P\cdot\ddot{s}_{\overline{20}|.07} \Rightarrow X = \frac{P\,\ddot{s}_{\overline{20}|.07}}{12(8.74)}$$

You are told the contract reserve at the end of year five is 10,000:

Retrospective reserve = Accumulated premiums - Accum benefits

$$10{,}000 = P\cdot\ddot{s}_{\overline{5}|.07} - \emptyset$$

$$P = 10{,}000 / \ddot{s}_{\overline{5}|.07}$$

Now substitute the value of P in the earlier formula, and solve for the value of X

$$X = \left(\frac{10{,}000}{\ddot{s}_{\overline{5}|.07}}\right)\left(\frac{\ddot{s}_{\overline{20}|.07}}{12(8.74)}\right) = \frac{10{,}000}{6.1533}\left(\frac{43.8652}{12(8.74)}\right)$$

$$= \frac{438{,}652}{645.3581}$$

$$= 679.70$$

E

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Added details 4/23/96

(7) This is the detailed derivation of the starting point for the solution on the prior page. This time diagram shows the premiums paid prior to age 65, and the death benefit at the end of each year:

	P	P	P	$\cdots$	P	P		
	45	46	47	$\cdots$	63	64	65	
Death benefit		$P\ddot{s}_{\overline{1}\rceil i}$	$P\ddot{s}_{\overline{2}\rceil i}$		$P\ddot{s}_{\overline{18}\rceil i}$	$P\ddot{s}_{\overline{19}\rceil i}$	$P\ddot{s}_{\overline{20}\rceil i}$	

$$P\ddot{a}_{45:\overline{20}\rceil} = 12 \times \ddot{a}^{(12)}_{65}\frac{D_{65}}{D_{45}} + P\left[\ddot{s}_{\overline{1}\rceil}\left(\frac{C_{45}}{D_{45}}\right) + \ddot{s}_{\overline{2}\rceil}\left(\frac{C_{46}}{D_{45}}\right) + \dots + \ddot{s}_{\overline{19}\rceil}\left(\frac{C_{63}}{D_{45}}\right) + \ddot{s}_{\overline{20}\rceil}\left(\frac{C_{64}}{D_{45}}\right)\right]$$

Now, simplify the final expression for pre-retirement death benefits

$$P\left(\ddot{s}_{\overline{1}\rceil} v q_{45} + \ddot{s}_{\overline{2}\rceil} v^2 p_{45} q_{46} + \dots + \ddot{s}_{\overline{19}\rceil} v^{19}\,{}_{18}p_{45} q_{63} + \ddot{s}_{\overline{20}\rceil} v^{20}\,{}_{19}p_{45} q_{64}\right]$$

$$= (P/d)\left[((1+i)^1-1)v q_{45} + ((1+i)^2-1)v^2 p_{45} q_{46} + \dots + ((1+i)^{19}-1)v^{19}\,{}_{18}p_{45} q_{63} + ((1+i)^{20}-1)v^{20}\,{}_{19}p_{45} q_{64}\right]$$

$$= (P/d)\left[(1-v)q_{45} + (1-v^2)p_{45}q_{46} + \dots + (1-v^{19})\,{}_{18}p_{45}q_{63} + (1-v^{20})\,{}_{19}p_{45}q_{64}\right]$$

$$= \left(\frac{P}{d}\right)\left[\begin{array}{l} q_{45} + p_{45}q_{46} + \dots + {}_{18}p_{45}q_{63} + {}_{19}p_{45}q_{64} \\ -vq_{45} - v^2 p_{45} q_{46} - \dots - v^{19}\,{}_{18}p_{45}q_{63} - v^{20}\,{}_{19}p_{45}q_{64} \end{array}\right]$$

$$= \left(\frac{P}{d}\right)\left[\frac{d_{45}}{l_{45}} + \frac{l_{46}}{l_{45}}\left(\frac{d_{46}}{l_{46}}\right) + \dots + \frac{l_{63}}{l_{45}}\left(\frac{d_{63}}{l_{63}}\right) + \frac{l_{64}}{l_{45}}\left(\frac{d_{64}}{l_{64}}\right) - A^{1}_{45:\overline{20}\rceil}\right]$$

$$= (P/d)\left({}_{20}q_{45} - A^{1}_{45:\overline{20}\rceil}\right)$$

Now plug this back into the original expression

$$P\,\ddot{a}_{45:\overline{20}\rceil} = 12 \times \ddot{a}^{(12)}_{65}(D_{65}/D_{45}) + (P/d)\left({}_{20}q_{45} - A^{1}_{45:\overline{20}\rceil}\right)$$

Now substitute $\ddot{a}_{45:\overline{20}\rceil} = \dfrac{1 - A_{45:\overline{20}\rceil}}{d}$

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(7) $(P/d)(1-A_{45:\overline{20|}}) = 12 \times \ddot{a}^{(12)}_{65} \frac{D_{65}}{D_{45}} + (P/d)({}_{20}q_{45} - A^{1}_{45:\overline{20|}})$

$(P/d)(1-A_{45:\overline{20|}} - {}_{20}q_{45} + A^{1}_{45:\overline{20|}}) = 12 \times \ddot{a}^{(12)}_{65} D_{65}/D_{45}$

$\frac{P}{d}(1 - {}_{20}q_{45} - A^{1}_{45:\overline{20|}}) = 12 \times \ddot{a}^{(12)}_{65} \frac{D_{65}}{D_{45}}$

$\frac{P}{d}({}_{20}p_{45} - v^{20}\,{}_{20}q_{45}) = 12 \times \ddot{a}^{(12)}_{65}(v^{20}\,{}_{20}q_{45})$

$\frac{P}{d}({}_{20}p_{45})(1-v^{20}) = 12 \times \ddot{a}^{(12)}_{65}(v^{20}\,{}_{20}q_{45})$

$P\left(\frac{1-v^{20}}{d}\right) = 12 \times \ddot{a}^{(12)}_{65}\, v^{20}$

$P\,\ddot{s}_{\overline{20|}} = 12 \times \ddot{a}^{(12)}_{65}$

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8. Modified duration is defined as regular duration over $(1+i)$. Regular duration is a weighted average of the year of payment, where the weight is the present value of the payment:

$$\text{modified duration} = \frac{\bar{d}}{(1+i)} \qquad \bar{d} = \sum_{t=1}^{n} t v^t R_t \Big/ \sum_{t=1}^{n} v^t R_t$$

First you should draw a time diagram for the payments. Then you can write down the formula for the modified duration:

$$\frac{\bar{d}}{1+i} = \frac{1 \cdot v \cdot 80 + 2 \cdot v^2 (80+1000) + 10 v^{10} (1000)}{1.09 [v \cdot 80 + v^2 (80+1000) + v^{10} (1000)]}$$

$$= \frac{73.39 + 1818.03 + 4224.11}{1.09 (73.39 + 909.01 + 4224.11)}$$

$$= \frac{6115.53}{1.09 (1404.82)}$$

$$= 3.99$$

C

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9 This is another gross premium problem, similar to #4, but for an annuity. The key is to reflect each benefit and expense:

PV of gross premiums	$P \cdot \ddot{a}_{55:\overline{10}\rvert}$
First year commission	$.6P$
Subsequent commissions	$.1P\, a_{55:\overline{9}\rvert}$
Expenses	$2(12)\,{}_{10\rvert}\ddot{a}_{55}^{(12)}$
PV of annuity	$1200\,{}_{10\rvert}\ddot{a}_{55}^{(12)}$

The only simplification in this problem is to reflect the commission as 10% of P for 10 years, with an extra 50% in the first year:

$$P \cdot \ddot{a}_{55:\overline{10}\rvert} = .5P + .1P\,\ddot{a}_{55:\overline{10}\rvert} + (24 + 1200)\,{}_{10\rvert}\ddot{a}_{55}^{(12)}$$

$$P = \frac{1224\,(N_{65}^{(12)}/D_{55})}{.9\,\ddot{a}_{55:\overline{10}\rvert} - .5}$$

$$= \frac{1224\,(N_{65} - \frac{11}{24}D_{65})/D_{55}}{.9\,(N_{55} - N_{65})/D_{55} - .5}$$

$$= \frac{1224\,(N_{65} - \frac{11}{24}D_{65})}{.9\,(N_{55} - N_{65}) - .5\,D_{55}}$$

$$= \frac{1224\,(10{,}600 - \frac{11}{24}(1{,}000))}{.9\,(24{,}000 - 10{,}600) - .5(1{,}600)}$$

$$= \frac{12{,}413{,}400}{11{,}260}$$

$$= 1{,}102.43$$

A

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10. This is a surprisingly easy question. Since the decedent has died at 1/1/96, the charity's interest in the fund is the excess of the 1,000,000 paid to the charity over the present value of the annuity payable to the spouse each December 31:

$$PV \text{ of spouse annuity} = 70{,}000\, a_{65}$$
$$= 70{,}000 \frac{N_{66}}{D_{65}}$$
$$= 70{,}000 \left(\frac{773{,}638}{94{,}414}\right)$$
$$= 573{,}587$$

$$\text{Charity's interest} = 1{,}000{,}000 - 573{,}587$$
$$= 426{,}413$$

D

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11 The first step is to draw a time diagram for the deposits to the account:

Deposits	$12{,}000$	$12{,}000(1.04)$	$\cdots$	$12{,}000(1.04)^n$	$30{,}000$	$\cdots$	$30{,}000$	
	1/97	1/98					1/2025	
Compensation	$60{,}000(1.04)^1$	$60{,}000(1.04)^2$		$60{,}000(1.04)^{n+1}$			$60{,}000(1.04)^{29}$	

The key point of the problem is that the deposit will be limited to 30,000 at a future date, when $12{,}000(1.04)^n > 30{,}000$. Solve for n by setting them equal to each other:

$$12{,}000(1.04)^n = 30{,}000 \Rightarrow (1.04)^n = 2.5$$
$$n \log 1.04 = \log 2.5$$
$$n = (\log 2.5)/(\log 1.04)$$
$$= 23.36$$

When $n = 23$, the value of $12{,}000(1.04)^n$ should be less than 30,000. It is equal to 29,577. Now rewrite the time diagram, and calculate the P Value

Deposits	$12{,}000$	$12{,}000(1.04)^1$	$\cdots$	$12{,}000(1.04)^{23}$	$30{,}000$	$\cdots$	$30{,}000$	AV
	1/97	1/98		1/2020	1/2021		1/2025	

$$AV = 12{,}000(1.07)^{28} + 12{,}000(1.04)^1(1.07)^{27} + \ldots + 12{,}000(1.04)^{23}(1.07)^5 + 30{,}000\, s_{\overline{5}|.07}$$

Note that the sum of the exponents is 28 for the 12,000 terms

$$AV = (1.04)^{28}(12{,}000)\left[\left(\frac{1.07}{1.04}\right)^{28} + \left(\frac{1.07}{1.04}\right)^{27} + \cdots + \left(\frac{1.07}{1.04}\right)^{5}\right] + 30{,}000\, s_{\overline{5}|.07}$$
$$= (1.04)^{28}(12{,}000)\left(s_{\overline{29}|2.88\%} - s_{\overline{5}|2.88\%}\right) + 30{,}000\, s_{\overline{5}|.07}$$
$$= (2.9987)(12{,}000)(39.1179) + 30{,}000(5.7507)$$
$$= 1{,}580{,}158$$

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12 There are many approaches to working this type of problem. They all test your knowledge of various identities between various commutation functions.

You are given information about both an increasing insurance and an increasing annuity:

$$8{,}500 = 1{,}000(IA)_{65} \qquad 40{,}000 = 500(I\ddot{a})_{65}$$
$$8.5 = (IA)_{65} = R_{65}/D_{65} \qquad 80 = (I\ddot{a})_{65} = S_{65}/D_{65}$$

The quickest way to determine the value of S_{66} is to use this insurance identity:

$$M_x = D_x - d \cdot N_x$$
$$R_x = N_x - d \cdot S_x = S_x - S_{x+1} - (1-v)S_x = vS_x - S_{x+1}$$
$$R_{65} = vS_{65} - S_{66}$$
$$S_{66} = vS_{65} - R_{65}$$

Now you have various information based on R_{65}, S_{65}, and D_{65}. You can use the l_{65} value and the interest rate to calculate D_{65}, and get the answer:

$$\begin{aligned} S_{66} &= v(80D_{65}) - (8.5D_{65}) \\ &= D_{65}(80v - 8.5) \\ &= v^{65} l_{65}(80v - 8.5) \\ &= (1.05)^{-65}(6800)(67.69) \\ &= 19{,}307.77 \end{aligned}$$

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An alternative approach based on a different obscure identity using increasing insurances is shown on the next page

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(12) Based on the information given, you can write down formulas for the increasing insurance and the annuity

Insurance	Annuity
$8500 = 1000\, IA_{65}$	$40{,}000 = 500\, (I\ddot{a})_{65}$
$8.5 = IA_{65} = \frac{R_{65}}{D_{65}}$	$80 = (I\ddot{a})_{65} = \frac{S_{65}}{D_{65}}$

Using the values of l_{65} and l_{66}, you can calculate D_{65} and D_{66}. But that doesn't help you calculate $S_{66} = S_{65} - N_{65}$. You need to calculate a value for N_{65}.

The key to the solution is the identity for the increasing insurance, which is similar to that for a level insurance:

$$A_x = 1 - d\,\ddot{a}_x \qquad IA_x = \ddot{a}_x - d(I\ddot{a})_x$$

$$\ddot{a}_x = IA_x + d(I\ddot{a})_x$$

$$\ddot{a}_{65} = 8.5 + (.05/1.05)80$$

$$= 12.3095$$

$$\frac{S_{66}}{D_{66}} = (I\ddot{a})_{66} = \frac{S_{65} - N_{65}}{D_{66}} = \frac{D_{65}}{D_{66}}\left(\frac{S_{65}}{D_{65}} - \frac{N_{65}}{D_{65}}\right)$$

$$S_{66} = D_{65}\left((I\ddot{a})_{65} - \ddot{a}_{65}\right)$$

$$= v^{65} l_{65}\,(80 - 12.3095)$$

$$= (1.05)^{-65}(6{,}800)(67.6905)$$

$$= 19{,}307.77$$

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Solution revised 2-19-2000

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13 With a callable bond, you should calculate (or write down formulas for) prices based on all possible redemption dates. One of the quicker price calculation formulas is the Alternate formula:

$$\text{Price} = C + (Fr - Ci)\, a_{\overline{n}|i}$$

C = redemption
F = Face
r = coupon rate
i = yield rate

Redemption Date	
12/31/12	$P = 1050 + (1000(.04) - 1050(.05))\, a_{\overline{17}|.05}$ $= 1050 - 12.5\, a_{\overline{17}|.05}$ $= 1050 - 12.5(11.2741)$ $= 909.07$
12/31/13	$P = 1025 + (1000(.04) - 1025(.05))\, a_{\overline{18}|.05}$ $= 1025 - 11.25\, a_{\overline{18}|.05}$ $= 893.49$
12/31/14	$P = 1010 + (1000(.04) - 1010(.05))\, a_{\overline{19}|.05}$ $= 1010 - 10.5\, a_{\overline{19}|.05}$ $= 883.10$
12/31/15	$P = 1025 + (1000(.04) - 1025(.05))\, a_{\overline{20}|.05}$ $= 1025 - 11.25\, a_{\overline{20}|.05}$ $= 884.80$

The lowest price for all four dates is 883.10. It was important to look at the original maturity date, not just the earlier call dates!

A

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14 This is a tricky problem for two reasons. One is that they give you the value for q^s_{41}, but you never use it in the problem due to the preferred mortality definition.

The second tricky aspect is that the preferred mortality rate is defined as $v \cdot q^s_x$. When you plug these values into your formulas, it gets very confusing because the v^n terms no longer make any sense!

From Smith's information, you have

$12{,}000 = 100{,}000\, A_{42}$ for standard mortality

For Brown's policy, you have

$$X = 100{,}000\left(v\, q^p_{[40]} + v^2\, {}_{p}p^p_{[40]}\, q^p_{[40]+1} + v^3\, {}_{2}p^p_{[40]}\, q^s_{42} + \ldots\right)$$

$$= 100{,}000\left(v\, q^p_{[40]} + v^2\left(1-q^p_{[40]}\right)\left(q^p_{[40]}\right) + v^2\left(1-q^p_{[40]}\right)^2 A^s_{42}\right)$$

In this expression, the select period is gone starting at age 42, and $q^p_{[40]+1} = q^p_{[40]}$. Now, plug in the formula:

$$X = 100{,}000\left(v\left(v\, q^s_{40}\right) + v^2\left(1-\left(v\, q^s_{40}\right)\right)\left(v\, q^s_{40}\right) + v^2\left(1-\left(v\, q^s_{40}\right)\right)^2 A^s_{42}\right)$$

$$= 100{,}000\left(\frac{.01}{(1.08)^2} + \frac{(1-.01/1.08).01}{(1.08)^3} + \frac{(1-.01/1.08)^2}{(1.08)^2}\left(\frac{12}{100}\right)\right)$$

$$= 857.34 + 786.48 + 10{,}098$$

$$= 11{,}742.25$$

B

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15. This is a slightly unusual problem with both complete and curtate expectation of life.

Complete $\mathring{e}_x \doteq$ Curtate $e_x + \frac{1}{2}$

What you should do is set up a table to calculate each year's expectation of life, the account balance, and the annuity payment.

$e_{71} \doteq \mathring{e}_{71} - \frac{1}{2} = 14.80$

$e_x = {}_1p_x + {}_2p_x + {}_3p_x + \ldots$ (annuity immediate at 0% i)

$e_{x+1} = {}_1p_{x+1} + {}_2p_{x+1} + {}_3p_{x+1} + \ldots$

$= (l_{x+2} + l_{x+3} + l_{x+4} + \ldots)/l_{x+1}$

$e_{x+1} = e_x\left(\frac{l_x}{l_{x+1}}\right) - 1 = \frac{e_x}{p_x} - 1$

The formula above is necessary to calculate future e_x values:

Date	Age x	e_x	p_x	Account balance before payment	Acct bal / e_x = Payment	EOY account balance = (Acct Bal - Payment) 1.07
1-1-95	71	14.800	.98	100,000	6,756.76	1.07(100,000 - 6757)
1-1-96	72	14.102	.97	99,770	7,075	1.07(99,770 - 7,075)
1-1-97	73	13.538		99,184	7,326	

You only need to calculate the payment as of 1/1/97

D

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16. The first step is to draw a time diagram for this series of payments:

One way to work this problem is to use the given pattern of payments. It might be simpler to replace the "additional" payments of N every three years with three payments annually of Q that end at the same point in time:

Now you can "easily" set up expressions for the value of future payments at 1/1/94 and 1/1/97

1/1/94 $P = N(a_{\overline{15}|.07}) + Q(\ddot{a}_{\overline{15}|.07})$ $\qquad Q\,s_{\overline{3}|.07} = N$

1/1/97 $P - 11{,}061 = N(\ddot{a}_{\overline{13}|.07}) + Q(\ddot{a}_{\overline{12}|.07})$

Now replace Q by $N / s_{\overline{3}|.07}$, and substitute the value of P in the second equation from the first

$$P = N(a_{\overline{15}|.07} + \ddot{a}_{\overline{15}|.07}/s_{\overline{3}|.07})$$

$$N\left(a_{\overline{15}|.07} + \frac{\ddot{a}_{\overline{15}|.07}}{s_{\overline{3}|.07}}\right) - 11{,}061 = N\left(\ddot{a}_{\overline{13}|.07} + \frac{\ddot{a}_{\overline{12}|.07}}{s_{\overline{3}|.07}}\right)$$

$$N = 11{,}061 \Big/ \left(a_{\overline{15}|.07} + \frac{\ddot{a}_{\overline{15}|.07}}{s_{\overline{3}|.07}} - \left(\ddot{a}_{\overline{13}|.07} + \frac{\ddot{a}_{\overline{12}|.07}}{s_{\overline{3}|.07}}\right)\right)$$

$$= 11{,}061 / (12.1393 - 11.5862)$$

$$= 20{,}000$$

E

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17. Smith's benefit can be expressed as a reversionary annuity. If the annuity started payments upon immediate death, it would be calculated as $(\ddot{a}_x^{(12)} - \ddot{a}_{xy}^{(12)})1200$.

Since no payments will be made for the first 10 years, and both Smith and his spouse must be alive 10 years from now for any payments to be made, it is relatively easy to write the expression for this deferred reversionary annuity:

$$\left(\frac{D_{35:35}}{D_{25:25}}\right)\left(\ddot{a}_{35}^{(12)} - \ddot{a}_{35:35}^{(12)}\right)1200$$

Now you can plug in the various commutation functions:

$$1200\left(\frac{8{,}807}{17{,}724}\right)\left(\frac{120{,}739}{9{,}031} - \frac{111{,}292}{8{,}807}\right) = 1200\left(\frac{D_{35:35}}{D_{25:25}}\right)\left(\frac{N_{35}^{(12)}}{D_{35}} - \frac{N_{35:35}^{(12)}}{D_{35:35}}\right)$$

$$= 1200(.4969)(13.3694 - 12.6368)$$
$$= 436.85$$

A

This question can be much more difficult when annuities involve certain periods or where more than two lives are considered (1994 #19).

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18. This problem is much easier that it looks at first. The increases in both the insurance benefit and the premium payments are canceled out by the interest rate.

$e_x = p_x + {}_2p_x + {}_3p_x + \cdots$ (immediate annuity at 0% i)

$$\text{PV of premiums} = X\left(1 + v p_{65}(1.05) + v^2 {}_2p_{65}(1.05)^2 + \ldots\right)$$
$$= X\left(1 + p_{65} + {}_2p_{65} + \ldots\right)$$
$$= X\left(1 + e_{65}\right)$$

$$\text{PV of death benefits} = 1000\left(v q_{65} + v^2 p_{65} q_{66}(1.05) + \ldots\right)$$
$$= \frac{1000}{1.05}\left(q_{65} + p_{65} q_{66} + {}_2p_{65} q_{67} + \ldots\right)$$
$$= \frac{1000}{1.05}\left(1 - p_{65} + p_{65}(1 - p_{66}) + p_{65} p_{66}(1 - p_{67}) + \ldots\right)$$
$$= \frac{1000}{1.05}\left(1 \underbrace{- p_{65} + p_{65} - p_{65}p_{66} + p_{65}p_{66} - p_{65}p_{66}p_{67} + \ldots}_{\text{telescoping sum} = \emptyset}\right)$$
$$= \frac{1000}{1.05}$$

Finally, you can now calculate the value of X

$$\text{PV of premiums} = \text{PV of death benefits}$$
$$X(1 + e_{65}) = 1000/1.05$$
$$X = 1000/[(1.05)(16)]$$
$$= 59.52$$

B

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19. The first step is to draw a time diagram for the monthly payments:

It is much more convenient to work with annual payments which increase once per annum, than it is to work with monthly payments that increase once per annum.

If you replace twelve monthly payments with a single payment each 12/31, then you have this time diagram for 15 annual payments:

$$100{,}000 = vZ + v^2(1.03)Z + \dots + v^{15}(1.03)^{14}Z$$

$$= vZ\left(1 + \frac{1.03}{1.09} + \dots + \left(\frac{1.03}{1.09}\right)^{14}\right)$$

$$\frac{1.09}{1.03} = 1.0583$$

$$Z = \frac{100{,}000(1.09)}{\ddot{a}_{\overline{15}|5.83\%}}$$

$$= 109{,}000/10.3964$$

$$= 10{,}484$$

1996 monthly payment $X = \dfrac{Z}{s_{\overline{12}|.72\%}}$ $\qquad (1.09)^{\frac{1}{12}} = 1.0072$

$$= 10{,}484/12.4874$$

$$= 839.56$$

2000 monthly payment $X(1.03)^4 = 839.56(1.03)^4$

$$= 944.94$$

E

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20. This problem is fairly tricky due to the potential for confusion between the nominal and effective rates. It is also difficult to interpret the information and derive the effective return on this investment!

First, calculate the price for the annuity based on the 5% nominal rate, compounded quarterly. Draw a time diagram for payments:

1-1-96 $P = 1000\, \ddot{a}_{\overline{80}|j}$ $\quad 1+j = 1.0125$ since 5% compounded quarterly

$= 51{,}016.49$

Next, calculate the accumulated value of all payments at 1/1/2016 at 10% nominal rate, compounded semiannually:

1-1-2016 $AV = 1000\, \ddot{s}_{\overline{80}|k}$ $\quad (1+k)^2 = 1.05$ since 10% compounded semiannually

$= 1000\, \ddot{s}_{\overline{80}|2.47\%}$

$= 250{,}622.70$

The effective rate of return is the interest rate that accumulates the 51,016 to 250,623 over 20 years:

$(1+i)^{20} = 250{,}622.70/51{,}016.49 = 4.9126$

$1+i = 1.0828$

C

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21 This problem tests your knowledge of formulas to convert between the rates of decrement for single decrement tables ($q'^{(i)}_x$) and probabilities of decrement for multiple decrement tables ($q^{(i)}_x$).

Single decrement tables

Age	rate #1	rate #2
x	$q'^{(1)}_x$	$q'^{(2)}_x$

$$p^{(T)}_x = [1-q'^{(1)}_x][1-q'^{(2)}_x]$$

Multiple decrement table probabilities

Age	#1	#2
x	$q^{(1)}_x$	$q^{(2)}_x$

$$p^{(T)}_x = 1 - q^{(1)}_x - q^{(2)}_x$$

Jordan's life contingencies has the standard formulas equating rates and probabilities for a two decrement table:

$$q^{(1)}_x \doteq \frac{q'^{(1)}_x\left(1-\frac{1}{2}q'^{(2)}_x\right)}{1-\frac{1}{4}q'^{(1)}_x q'^{(2)}_x}$$

$$q'^{(1)}_x \doteq \frac{q^{(1)}_x}{1-\frac{1}{2}q^{(2)}_x}$$

Use the second formula to calculate $q^{(1)}_x$:

$$q^{(1)}_x = q'^{(1)}_x\left(1-\tfrac{1}{2}q^{(2)}_x\right) = .03(1-.05) = .0285$$

$$q^{(T)}_x = q^{(1)}_x + q^{(2)}_x = .1285$$

C

Solution revised 2-19-2000

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22 This is a typical serial bond problem. You have three bonds purchased at the same date, but with three different redemption dates. In some older problems on this exam, there would be multiple purchase dates <u>and</u> multiple redemption dates, since each bond had the same period to maturity.

The best formula to use for this serial bond problem is Makeham's formula:

$$P = K + \frac{g}{i}(C-K)$$

In general, most of the work for Makeham's formula is determining the value of K, which is the present value of the redemption amount. With the serial bond, you have three different redemption amounts, so Makeham's formula can be thought of this way

$$\Sigma P = \Sigma K + \frac{g}{i}(\Sigma C - \Sigma K)$$

The formula above represents the sum of Makeham's formula applied to each bond individually.

$$\Sigma K = 7000\left((1.0302)^{-4} + (1.0302)^{-8} + (1.0302)^{-12}\right) = 16{,}627$$

$$\frac{g}{i} = \frac{Fr}{Ci} = \frac{7000\,(\text{coupon rate})}{7000\,(\text{interest rate})} = \frac{.0250}{.0302}$$

Solution revised 2-19-2000

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(22) The interest rate of 6% compounded quarterly produces $(1.015)^2 = 1.0302$ on a semiannual basis.

Now you can calculate the price of the serial bond using makeham's formula:

$$\Sigma P = \Sigma K + \frac{g}{i}(\Sigma C - \Sigma K)$$

$$= 16{,}627 + \frac{.0250}{.0302}(7000(3) - 16{,}627)$$

$$= 20{,}244$$

(A)

You could use any of the bond price formulas to calculate the same result, but it is more work. In effect, you use the formula separately for each bond. Here is the result using the standard bond price formula:

$$P = 7{,}000\,(.025)\,a_{\overline{4}|3.02\%} + 7{,}000\,(1.0302)^{-4}$$
$$+ 7{,}000\,(.025)\,a_{\overline{8}|3.02\%} + 7{,}000\,(1.0302)^{-8}$$
$$+ 7{,}000\,(.025)\,a_{\overline{12}|3.02\%} + 7{,}000\,(1.0302)^{-12}$$
$$= 7{,}000\big(.025(3.8274/1.0302) + .8877$$
$$+ .025(7.2250/1.0302) + .7880$$
$$+ .025(10.2411/1.0302) + .6995\big)$$
$$= 20{,}244$$

Solution added 2-19-2000

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(23) There is a quicker way to work this problem, and that is to use annual (not monthly) payments for the refinanced loan. Once you know the outstanding loan balance is 83,862 at 1-1-96, proceed as follows

$$\text{Annual payment} = \frac{83862}{a_{\overline{18}|7.25\%}} = 8487.97$$

1-1-2003	O/S Balance	$8487.97(a_{\overline{11}|7.25\%})$ =	62,863
1-1-2002	O/S Balance	$8487.97(a_{\overline{12}|7.25\%})$ =	66,528
		Δ	3,665

It still is strange that annual versus monthly would produce the same result for principal repayment in any specific year.

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24. In 1966, the stationary population has 800,000 total lives. Half are under age 30, and half are over age 30. The population is sustained by annual births of 15,000.

In 1966, the annual birth rate increases to 16,000. If this continued indefinitely, the total population would increase to $800{,}000\left(\frac{16{,}000}{15{,}000}\right) = 853{,}333$, with half under age 30 and half over age 30.

At 1/1/96, the total population can be divided into two segments. One segment is everyone over age 30, which represents everyone who was under age 30 at 1-1-66. We know that there should still be 400,000 lives in that group, since they have now aged to the over 30 segment of the stationary population.

The other segment is the group under age 30, who are the results of the increased birth rate of 16,000 per year. We know that this group should be half of the 853,333 = 426,667 lives.

The total population at 1-1-96 is 400,000 over age 30 + 426,667 under = 826,667 total. (C)

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25. This is a simple identity. How do you calculate the value of life insurance based on q_x values?

$$A_{102} = v\,q_{102} + v^2 p_{102} q_{103} + v^3 p_{102} p_{103} q_{104} + \cdots$$

Actually, there are no more terms, since $q_{104} = 1.0$. This means that $p_{104} = 0$, and the rest of the series is also zero.

You need to expand on the l_x values you are given, and calculate values for p_x and q_x:

Age x	l_x	d_x	q_x	p_x
103	250	150	.6	.4
104	100	100	1.0	0
105	0	—		

$p_{102} = 1 - q_{102}$
$= .750$

$$A_{102} = \frac{.250}{1.07} + \frac{.75(.6)}{(1.07)^2} + \frac{.75(.4)(1.0)}{(1.07)^3}$$
$$= .2336 + .3930 + .2449$$
$$= .8715$$

B